Fluid Mechanics: Fundamentals and Applications
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Chapter 7

DIMENSIONAL ANALYSIS AND MODELING

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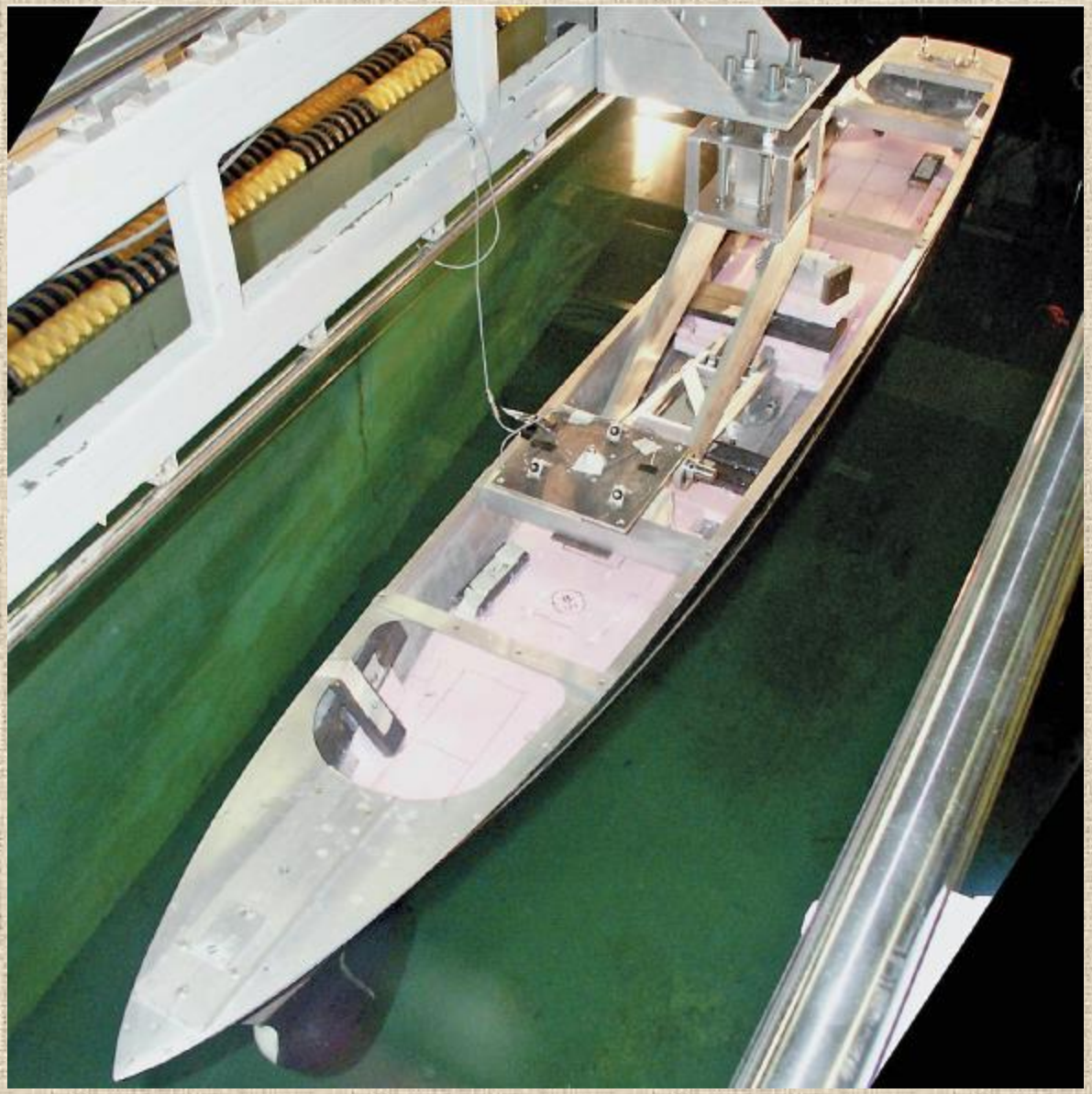


A 1:46.6 scale model of an Arleigh Burke class U.S. Navy fleet destroyer being tested in the 100-m long towing tank at the University of Iowa. The model is 3.048 m long. In tests like this, the Froude number is the most important nondimensional parameter.

Objectives

- Develop a better understanding of dimensions, units, and dimensional homogeneity of equations
- Understand the numerous benefits of dimensional analysis
- Know how to use the method of repeating variables to identify nondimensional parameters
- Understand the concept of dynamic similarity and how to apply it to experimental modeling

7–1 ■ DIMENSIONS AND UNITS

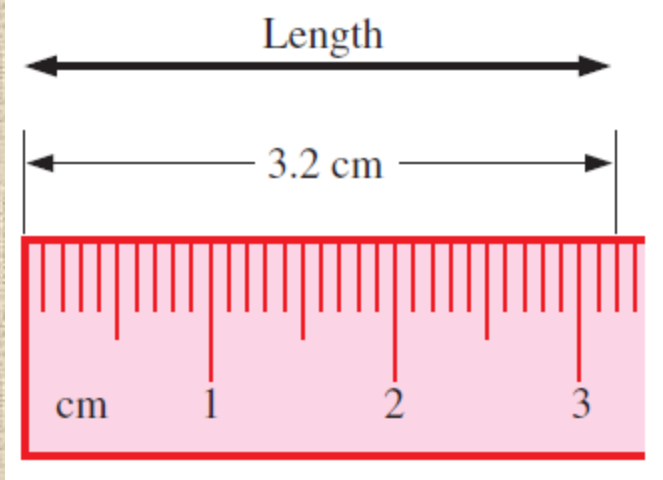
Dimension: A measure of a physical quantity (without numerical values).

Unit: A way to assign a *number* to that dimension.

There are seven **primary dimensions** (also called **fundamental** or **basic dimensions**): mass, length, time, temperature, electric current, amount of light, and amount of matter.

All nonprimary dimensions can be formed by some combination of the seven primary dimensions.

Dimensions of force: $\{\text{Force}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\right\} = \{\text{mL/t}^2\}$



A *dimension* is a measure of a physical quantity without numerical values, while a *unit* is a way to assign a number to the dimension. For example, length is a dimension, but centimeter is a unit.

Primary dimensions and their associated primary SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time†	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

EXAMPLE 7–1 Primary Dimensions of Surface Tension

An engineer is studying how some insects are able to walk on water (Fig. 7–2). A fluid property of importance in this problem is surface tension (σ_s), which has dimensions of force per unit length. Write the dimensions of surface tension in terms of primary dimensions.

SOLUTION The primary dimensions of surface tension are to be determined.

Analysis From Eq. 7–1, force has dimensions of mass times acceleration, or $\{mL/t^2\}$. Thus,

$$\textit{Dimensions of surface tension: } \{\sigma_s\} = \left\{\frac{\text{Force}}{\text{Length}}\right\} = \left\{\frac{m \cdot L/t^2}{L}\right\} = \{m/t^2\} \quad (1)$$

Discussion The usefulness of expressing the dimensions of a variable or constant in terms of primary dimensions will become clearer in the discussion of the method of repeating variables in Section 7–4.

The water strider is an insect that can walk on water due to surface tension.

7–2 ■ DIMENSIONAL HOMOGENEITY

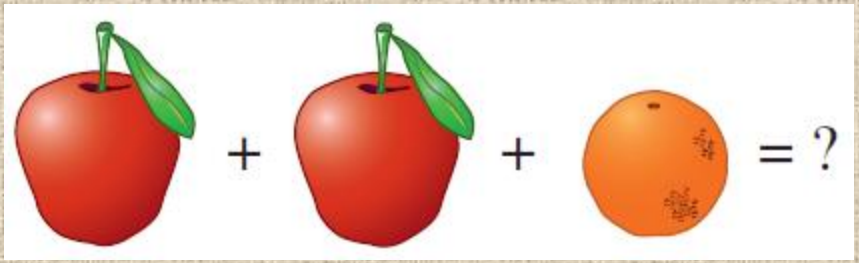
The law of dimensional homogeneity: Every additive term in an equation must have the same dimensions.

Change of total energy of a system: $\Delta E = \Delta U + \Delta \mathrm{KE} + \Delta \mathrm{PE}$

$$\Delta U = m(u_2 - u_1)$$

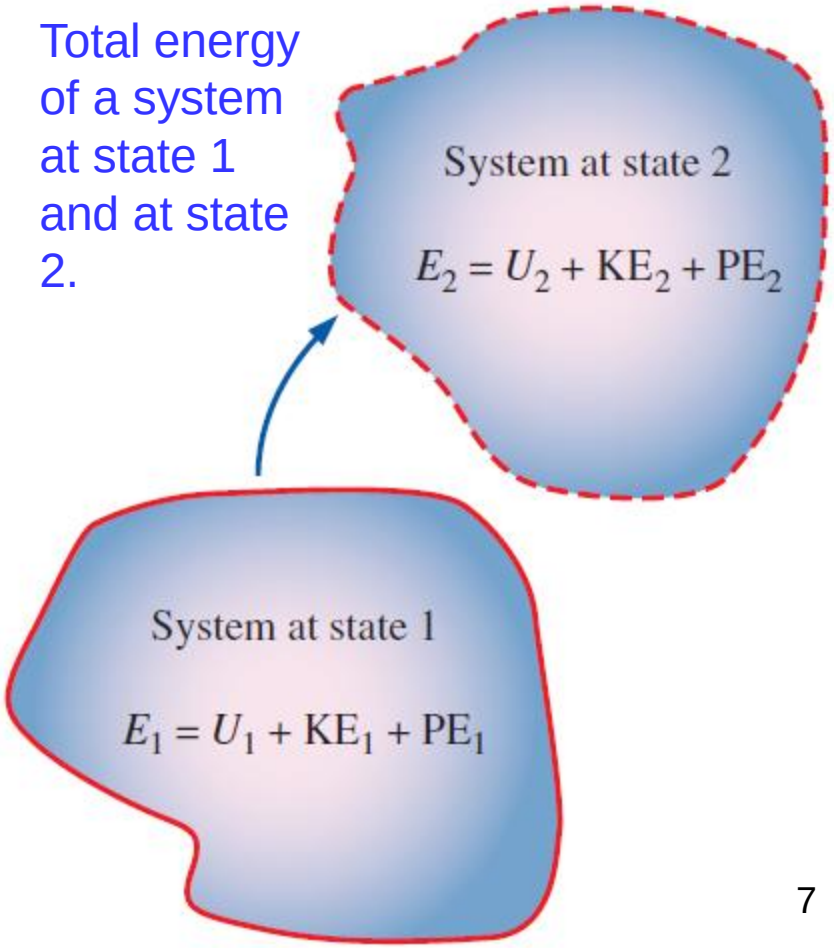
$$\Delta \mathrm{KE} = \frac{1}{2} m(V_2^2 - V_1^2)$$

$$\Delta \mathrm{PE} = mg(z_2 - z_1)$$



You can't add apples and oranges!

Total energy of a system at state 1 and at state 2.

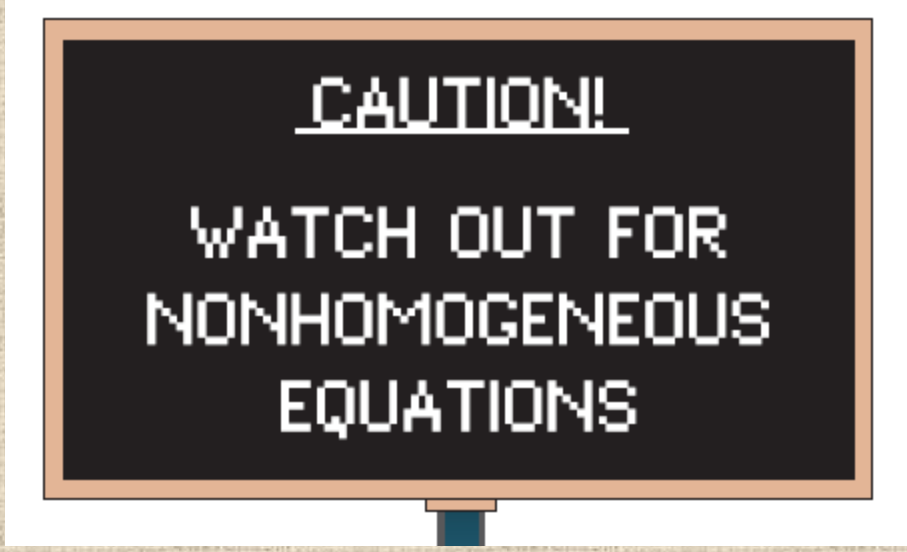


$$\{\Delta E\} = \{\text{Energy}\} = \{\text{Force} \cdot \text{Length}\} \quad \rightarrow \quad \{\Delta E\} = \{\text{mL}^2/\text{t}^2\}$$

$$\{\Delta U\} = \left\{\text{Mass}\,\frac{\text{Energy}}{\text{Mass}}\right\} = \{\text{Energy}\} \quad \rightarrow \quad \{\Delta U\} = \{\text{mL}^2/\text{t}^2\}$$

$$\{\Delta \text{KE}\} = \left\{\text{Mass}\,\frac{\text{Length}^2}{\text{Time}^2}\right\} \quad \rightarrow \quad \{\Delta \text{KE}\} = \{\text{mL}^2/\text{t}^2\}$$

$$\{\Delta \text{PE}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\,\text{Length}\right\} \quad \rightarrow \quad \{\Delta \text{PE}\} = \{\text{mL}^2/\text{t}^2\}$$



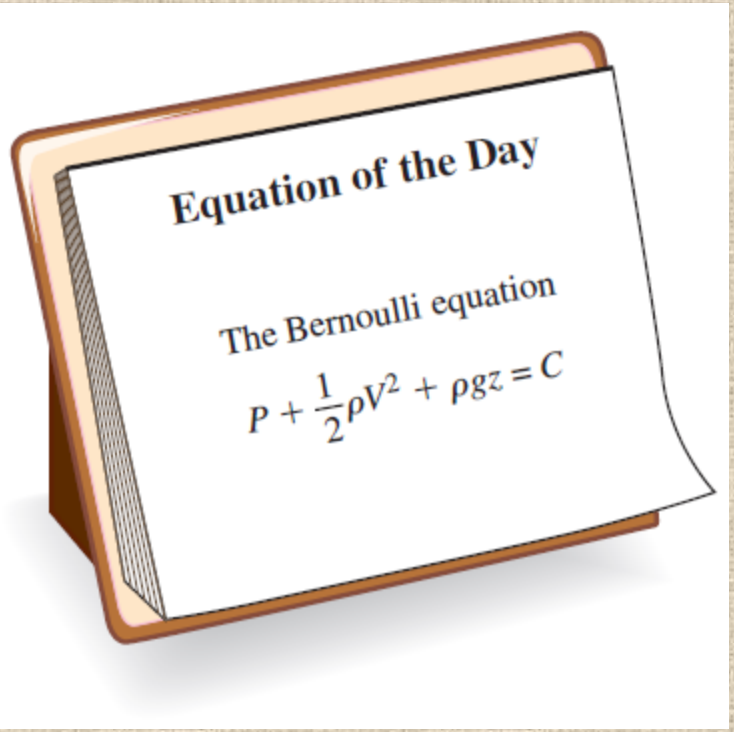
An equation that is not dimensionally homogeneous is a sure sign of an error.

EXAMPLE 7–2 Dimensional Homogeneity of the Bernoulli Equation

Probably the most well-known (and most misused) equation in fluid mechanics is the Bernoulli equation (Fig. 7–6), discussed in Chap. 5. One standard form of the Bernoulli equation for incompressible irrotational fluid flow is

Bernoulli equation:
$$P + \frac{1}{2}\rho V^2 + \rho g z = C \qquad (1)$$

(*a*) Verify that each additive term in the Bernoulli equation has the same dimensions. (*b*) What are the dimensions of the constant *C*?



The Bernoulli equation is a good example of a *dimensionally homogeneous* equation. All additive terms, including the constant, have the *same* dimensions, namely that of pressure. In terms of primary dimensions, each term has dimensions $\{m/(t^2L)\}$.

SOLUTION We are to verify that the primary dimensions of each additive term in Eq. 1 are the same, and we are to determine the dimensions of constant C.

Analysis (*a*) Each term is written in terms of primary dimensions,

$$\{P\} = \{\text{Pressure}\} = \left\{\frac{\text{Force}}{\text{Area}}\right\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\,\frac{1}{\text{Length}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\left\{\frac{1}{2}\rho V^2\right\} = \left\{\frac{\text{Mass}}{\text{Volume}}\left(\frac{\text{Length}}{\text{Time}}\right)^2\right\} = \left\{\frac{\text{Mass} \times \text{Length}^2}{\text{Length}^3 \times \text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\{\rho g z\} = \left\{\frac{\text{Mass}}{\text{Volume}}\,\frac{\text{Length}}{\text{Time}^2}\,\text{Length}\right\} = \left\{\frac{\text{Mass} \times \text{Length}^2}{\text{Length}^3 \times \text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

Indeed, **all three additive terms have the same dimensions.**

(*b*) From the law of dimensional homogeneity, the constant must have the same dimensions as the other additive terms in the equation. Thus,

Primary dimensions of the Bernoulli constant: $\{C\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$

Discussion If the dimensions of any of the terms were different from the others, it would indicate that an error was made somewhere in the analysis.

Nondimensionalization of Equations

Nondimensional equation: If we divide each term in the equation by a collection of variables and constants whose product has those same dimensions, the equation is rendered **nondimensional**.

Normalized equatiion: If the nondimensional terms in the equation are of order unity, the equation is called **normalized**.

Each term in a nondimensional equation is dimensionless.

Nondimensional parameters: In the process of nondimensionalizing an equation of motion, **nondimensional parameters** often appear—most of which are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number).

This process is referred to by some authors as **inspectional analysis**.

The nondimensionalized Bernoulli equation

$$\frac{P}{P_\infty} + \frac{\rho V^2}{2P_\infty} + \frac{\rho g z}{P_\infty} = \frac{C}{P_\infty}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\{1\} \quad \{1\} \quad \{1\} \quad \{1\}$$

A *nondimensionalized* form of the Bernoulli equation is formed by dividing each additive term by a pressure (here we use P_∞). Each resulting term is *dimensionless* (dimensions of {1}).

Equation of motion: $$\frac{d^2z}{dt^2} = -g$$

Dimensional result: $$z = z_0 + w_0 t - \frac{1}{2} g t^2$$

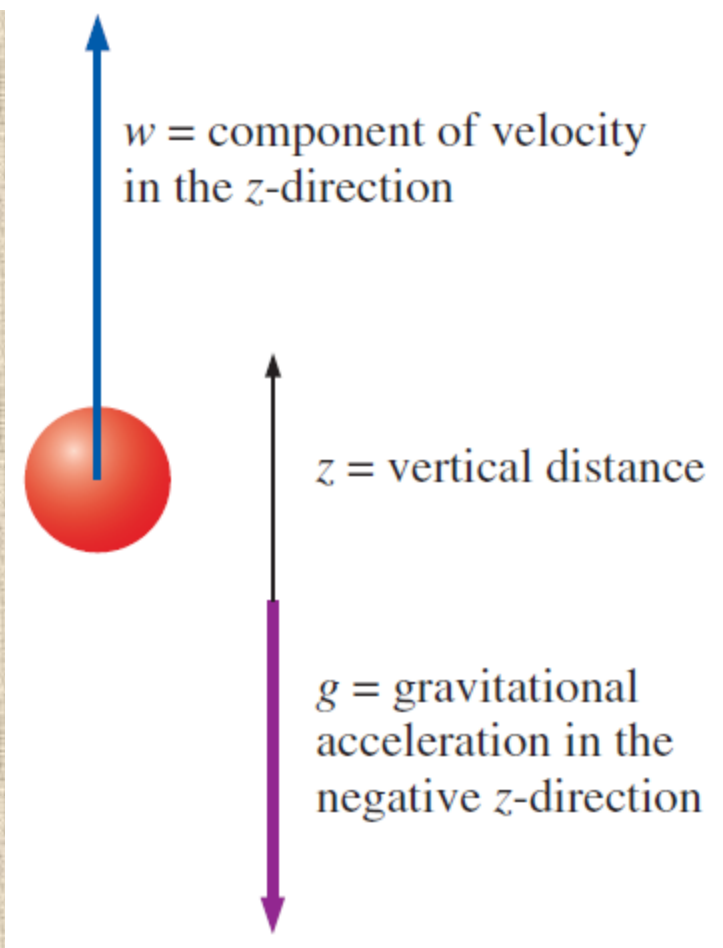
Dimensional variables: Dimensional quantities that change or vary in the problem. Examples: *z* (dimension of length) and *t* (dimension of time).

Nondimensional (or **dimensionless**) **variables:** Quantities that change or vary in the problem, but have no dimensions. Example: Angle of rotation, measured in degrees or radians, dimensionless units.

Dimensional constant: Gravitational constant *g*, while dimensional, remains constant.

Parameters: Refer to the combined set of dimensional variables, nondimensional variables, and dimensional constants in the problem.

Pure constants: The constant 1/2 and the exponent 2 in equation. Other common examples of pure constants are π and *e*.



Object falling in a vacuum. Vertical velocity is drawn positively, so $w < 0$ for a falling object.

To nondimensionalize an equation, we need to select **scaling parameters**, based on the primary dimensions contained in the original equation.

Primary dimensions of all parameters:

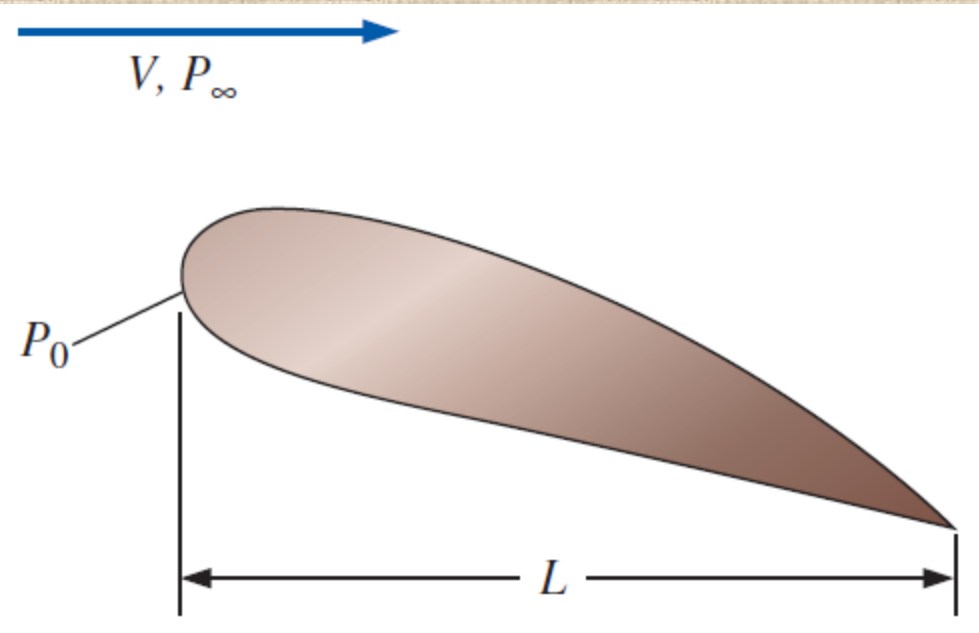
$$\{z\} = \{\mathrm{L}\} \qquad \{t\} = \{\mathrm{t}\} \qquad \{z_0\} = \{\mathrm{L}\} \qquad \{w_0\} = \{\mathrm{L/t}\} \qquad \{g\} = \{\mathrm{L/t^2}\}$$

Nondimensionalized variables: $z^* = \dfrac{z}{z_0} \qquad t^* = \dfrac{w_0 t}{z_0}$

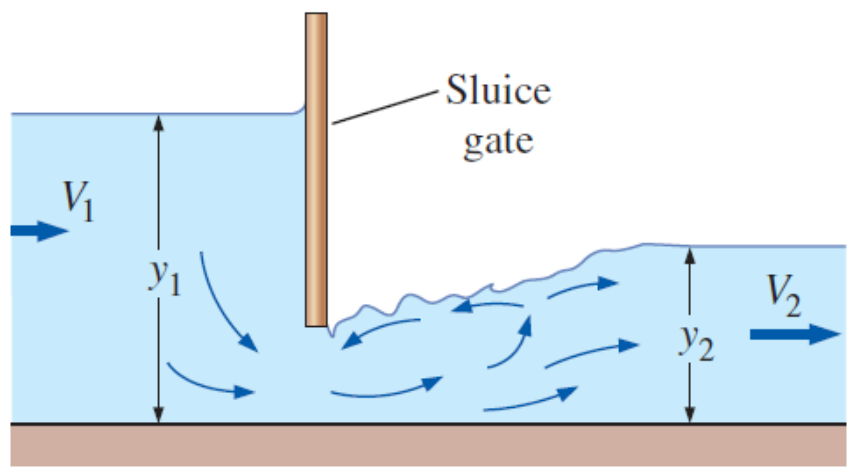
$$\frac{d^2z}{dt^2} = \frac{d^2(z_0 z^*)}{d(z_0 t^*/w_0)^2} = \frac{w_0^2}{z_0}\frac{d^2z^*}{dt^{*2}} = -g \quad \rightarrow \quad \frac{w_0^2}{gz_0}\frac{d^2z^*}{dt^{*2}} = -1$$

$$\mathrm{Fr} = \frac{w_0}{\sqrt{gz_0}}$$

Froude number



In a typical fluid flow problem, the scaling parameters usually include a characteristic length L, a characteristic velocity V, and a reference pressure difference $P_0 - P_\infty$. Other parameters and fluid properties such as density, viscosity, and gravitational acceleration enter the problem as well.



The *Froude number* is important in free-surface flows such as flow in open channels. Shown here is flow through a sluice gate. The Froude number upstream of the sluice gate is $\mathrm{Fr}_1 = V_1/\sqrt{gy_1}$, and it is $\mathrm{Fr}_2 = V_2/\sqrt{gy_2}$ downstream of the sluice gate.

Nondimensionalized equation of motion: $$\frac{d^2z^*}{dt^{*2}} = -\frac{1}{\mathrm{Fr}^2}$$

Nondimensional result: $$z^* = 1 + t^* - \frac{1}{2\mathrm{Fr}^2}t^{*2}$$

Relationships between key parameters in the problem are identified.

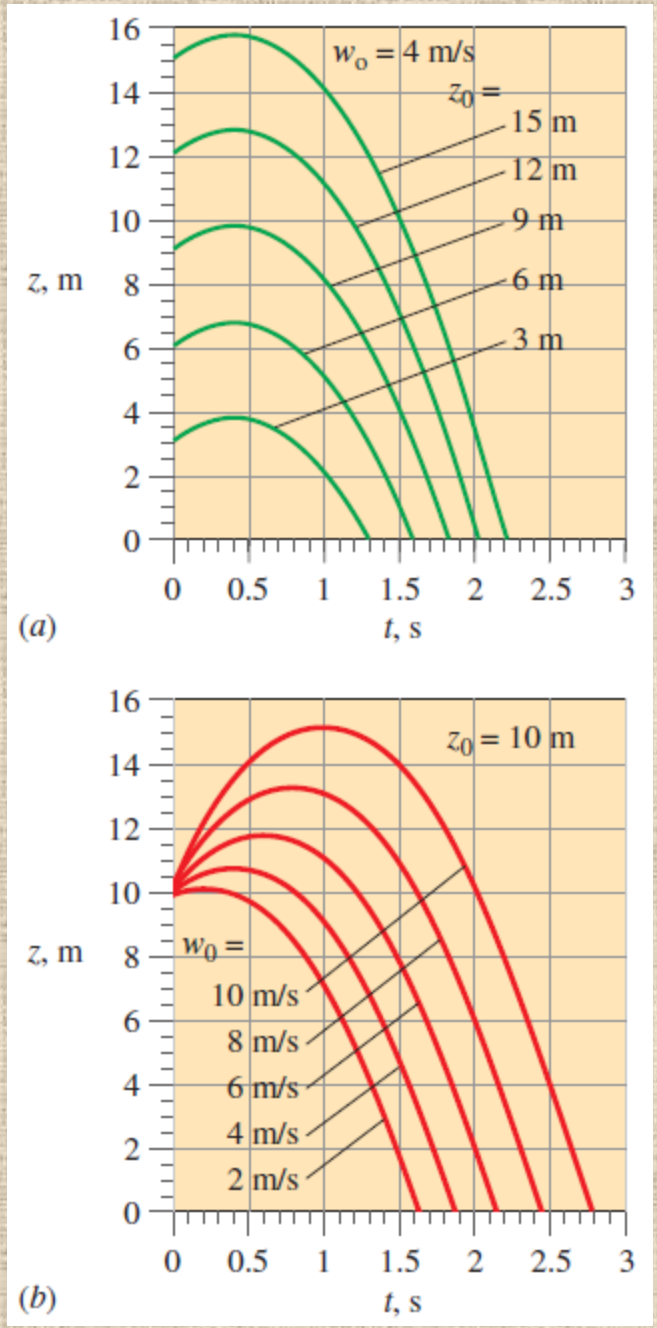
The number of parameters in a nondimensionalized equation is less than the number of parameters in the original equation.

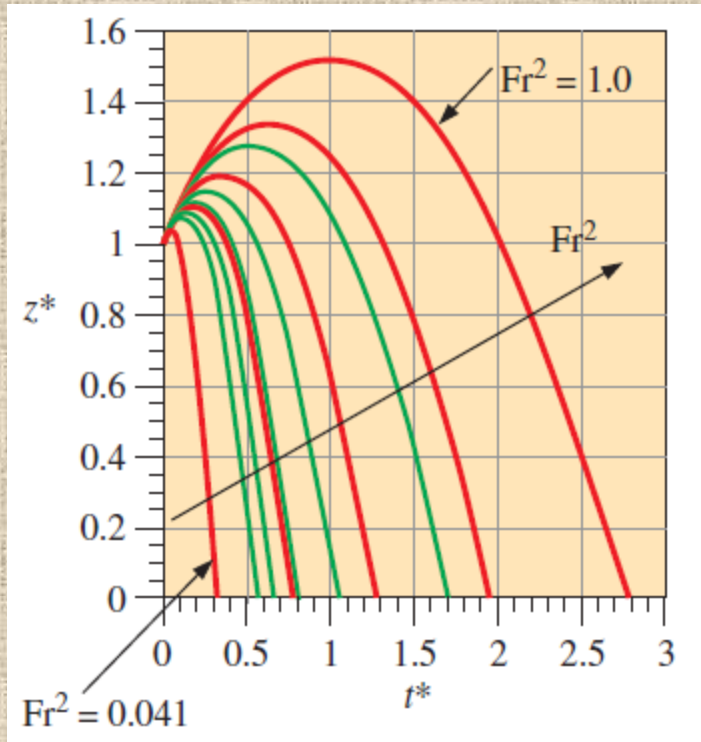
The two key advantages of nondimensionalization of an equation.

EXAMPLE 7–3 Illustration of the Advantages of Nondimensionalization

Your little brother's high school physics class conducts experiments in a large vertical pipe whose inside is kept under vacuum conditions. The students are able to remotely release a steel ball at initial height z_0 between 0 and 15 m (measured from the bottom of the pipe), and with initial vertical speed w_0 between 0 and 10 m/s. A computer coupled to a network of photosensors along the pipe enables students to plot the trajectory of the steel ball (height z plotted as a function of time t) for each test. The students are unfamiliar with dimensional analysis or nondimensionalization techniques, and therefore conduct several "brute force" experiments to determine how the trajectory is affected by initial conditions z_0 and w_0. First they hold w_0 fixed at 4 m/s and conduct experiments at five different values of z_0: 3, 6, 9, 12, and 15 m. The experimental results are shown in Fig. 7–12*a*. Next, they hold z_0 fixed at 10 m and conduct experiments at five different values of w_0: 2, 4, 6, 8, and 10 m/s. These results are shown in Fig. 7–12*b*. Later that evening, your brother shows you the data and the trajectory plots and tells you that they plan to conduct more experiments at different values of z_0 and w_0. You explain to him that by first nondimensionalizing the data, the problem can be reduced to just *one* parameter, and no further experiments are required. Prepare a nondimensional plot to prove your point and discuss.

Analysis Equation 7–4 is valid for this problem, as is the nondimensionalization that resulted in Eq. 7–9. As previously discussed, this problem combines three of the original dimensional parameters (g, z_0, and w_0) into *one* nondimensional parameter, the Froude number. After converting to the dimensionless variables of Eq. 7–6, the 10 trajectories of Fig. 7–12*a* and *b* are replotted in dimensionless format in Fig. 7–13. It is clear that all the trajectories are of the same family, with the Froude number as the only remaining parameter. Fr^2 varies from about 0.041 to about 1.0 in these experiments. If any more experiments are to be conducted, they should include combinations of z_0 and w_0 that produce Froude numbers outside of this range. A large number of additional experiments would be unnecessary, since all the trajectories would be of the same family as those plotted in Fig. 7–13.





Trajectories of a steel ball falling in a vacuum. Data of Fig. 7–12*a* and *b* are nondimensionalized and combined onto one plot.

Trajectories of a steel ball falling in a vacuum: (*a*) w_0 fixed at 4 m/s, and (*b*) z_0 fixed at 10 m (Example 7–3).

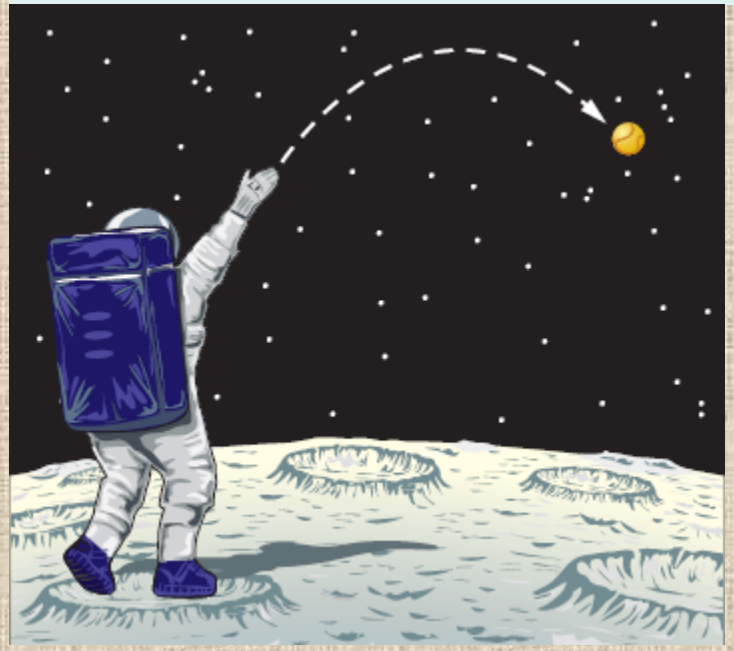
EXAMPLE 7–4 Extrapolation of Nondimensionalized Data

The gravitational constant at the surface of the moon is only about one-sixth of that on earth. An astronaut on the moon throws a baseball at an initial speed of 21.0 m/s at a 5° angle above the horizon and at 2.0 m above the moon's surface (Fig. 7–14). (*a*) Using the dimensionless data of Example 7–3 shown in Fig. 7–13, predict how long it takes for the baseball to fall to the ground. (*b*) Do an *exact* calculation and compare the result to that of part (*a*).

Properties The gravitational constant on the moon is $g_{moon} \cong 9.81/6 = 1.63\ \text{m/s}^2$.

Analysis (*a*) The Froude number is calculated based on the value of g_{moon} and the vertical component of initial speed,

$$w_0 = (21.0\ \text{m/s})\sin(5°) = 1.830\ \text{m/s}$$



Throwing a baseball on the moon

$$\mathrm{Fr}^2 = \frac{w_0^2}{g_{\text{moon}} z_0} = \frac{(1.830\ \text{m/s})^2}{(1.63\ \text{m/s}^2)(2.0\ \text{m})} = 1.03$$

This value of Fr^2 is nearly the same as the largest value plotted in Fig. 7–13. Thus, in terms of dimensionless variables, the baseball strikes the ground at $t^* \cong 2.75$, as determined from Fig. 7–13. Converting back to dimensional variables using Eq. 7–6,

Estimated time to strike the ground: $$t = \frac{t^* z_0}{w_0} = \frac{2.75(2.0\ \text{m})}{1.830\ \text{m/s}} = \mathbf{3.01\ s}$$

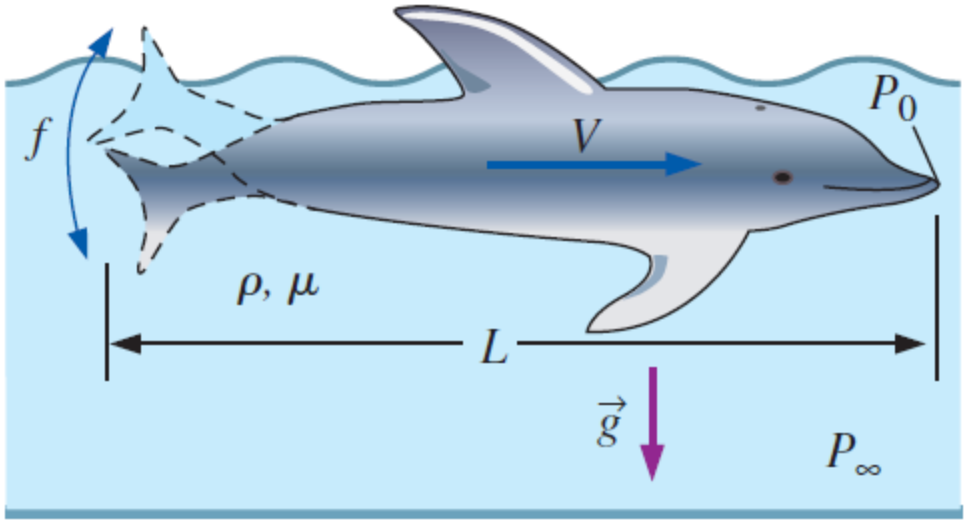
(*b*) An exact calculation is obtained by setting z equal to zero in Eq. 7–5 and solving for time t (using the quadratic formula),

Exact time to strike the ground:

$$t = \frac{w_0 + \sqrt{w_0^2 + 2z_0 g}}{g}$$

$$= \frac{1.830\ \text{m/s} + \sqrt{(1.830\ \text{m/s})^2 + 2(2.0\ \text{m})(1.63\ \text{m/s}^2)}}{1.63\ \text{m/s}^2} = \mathbf{3.05\ s}$$

Discussion If the Froude number had landed between two of the trajectories of Fig. 7–13, interpolation would have been required. Since some of the numbers are precise to only two significant digits, the small difference between the results of part (*a*) and part (*b*) is of no concern. The final result is $t = 3.0$ s to two significant digits.



$$\mathrm{Re} = \frac{\rho V L}{\mu}$$

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}}$$

$$\mathrm{St} = \frac{fL}{V}$$

$$\mathrm{Eu} = \frac{P_0 - P_\infty}{\rho V^2}$$

In a general unsteady fluid flow problem with a free surface, the scaling parameters include a characteristic length L, a characteristic velocity V, a characteristic frequency f, and a reference pressure difference $P_0 - P_\infty$. Nondimensionalization of the differential equations of fluid flow produces four dimensionless parameters: the Reynolds number, Froude number, Strouhal number, and Euler number (see Chap. 10).

7–3 ■ DIMENSIONAL ANALYSIS AND SIMILARITY

In most experiments, to save time and money, tests are performed on a geometrically scaled **model**, rather than on the full-scale **prototype**.

In such cases, care must be taken to properly scale the results. We introduce here a powerful technique called **dimensional analysis**.

The three primary purposes of dimensional analysis are

- To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results
- To obtain scaling laws so that prototype performance can be predicted from model performance
- To (sometimes) predict trends in the relationship between parameters

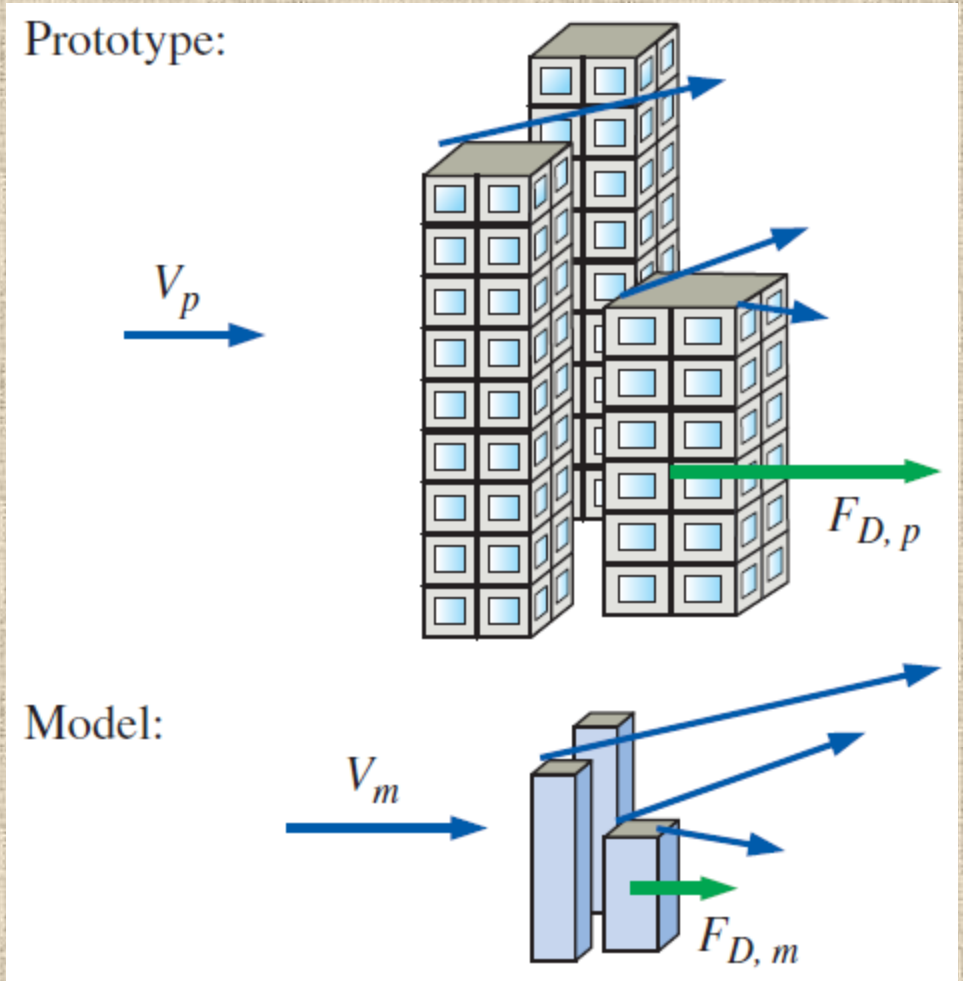
The principle of similarity

Three necessary conditions for complete similarity between a model and a prototype.

(1) Geometric similarity—the model must be the same shape as the prototype, but may be scaled by some constant scale factor.

(2) Kinematic similarity—the velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow.

(3) dynamic similarity—When all *forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow (*force-scale* equivalence).



Kinematic similarity is achieved when, at all locations, the speed in the model flow is proportional to that at corresponding locations in the prototype flow, and points in the same direction.

In a general flow field, complete similarity between a model and prototype is achieved only when there is geometric, kinematic, and dynamic similarity.

We let uppercase Greek letter Pi (Π) denote a nondimensional parameter.

In a general dimensional analysis problem, there is one Π that we call the **dependent** Π, giving it the notation Π_1.

The parameter Π_1 is in general a function of several other Π's, which we call **independent** Π's.

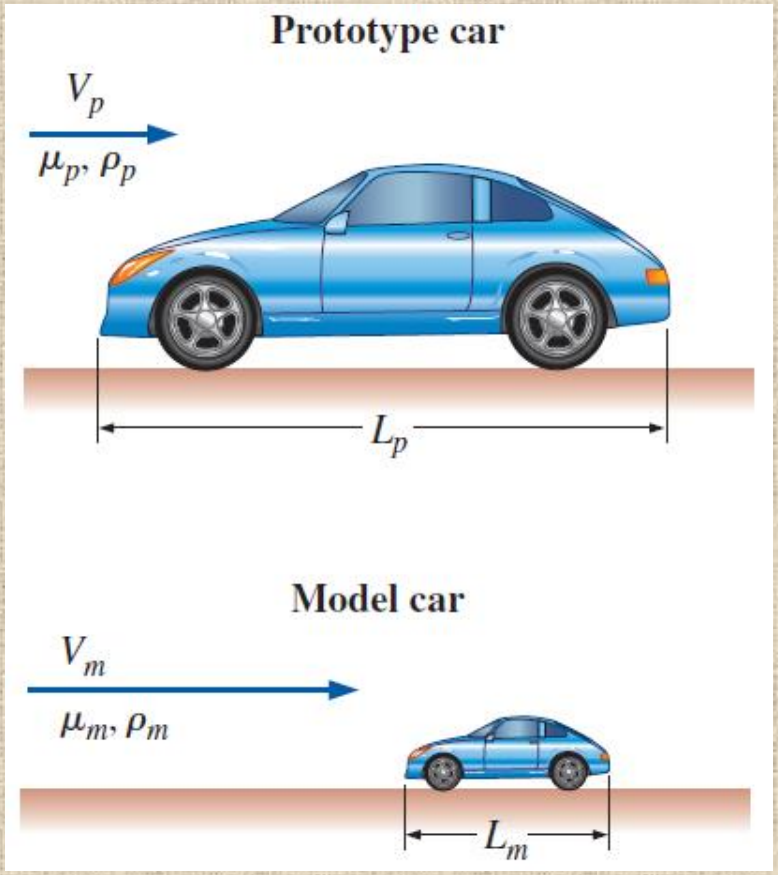
Functional relationship between Π*'s:* $\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_k)$

To ensure complete similarity, the model and prototype must be geometrically similar, and all independent groups must match between model and prototype.

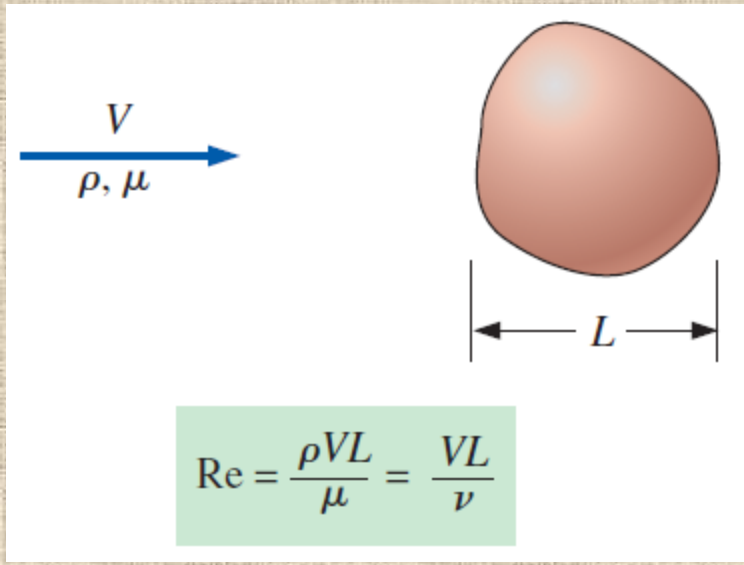
To achieve similarity

$$\text{If} \quad \Pi_{2,m} = \Pi_{2,p} \quad \text{and} \quad \Pi_{3,m} = \Pi_{3,p} \ldots \quad \text{and} \quad \Pi_{k,m} = \Pi_{k,p},$$

$$\text{then} \quad \Pi_{1,m} = \Pi_{1,p} \qquad (7\text{–}12)$$



Geometric similarity between a prototype car of length L_p and a model car of length L_m.



$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

The *Reynolds number* Re is formed by the ratio of density, characteristic speed, and characteristic length to viscosity. Alternatively, it is the ratio of characteristic speed and length to *kinematic viscosity*, defined as $\nu = \mu/\rho$.

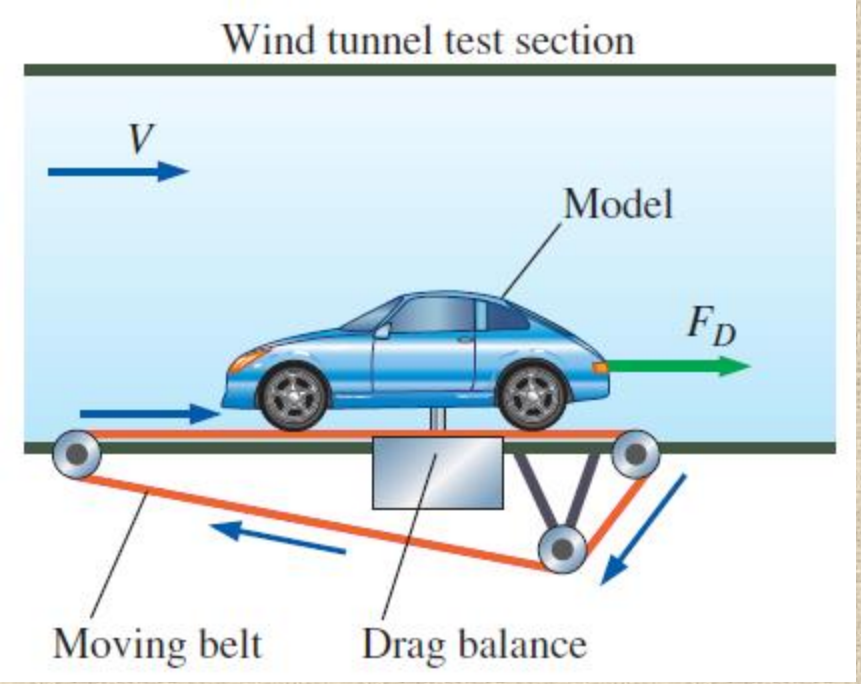
$$\Pi_1 = f(\Pi_2) \qquad \text{where} \qquad \Pi_1 = \frac{F_D}{\rho V^2 L^2} \qquad \text{and} \qquad \Pi_2 = \frac{\rho V L}{\mu}$$

The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics.

EXAMPLE 7–5 Similarity between Model and Prototype Cars

The aerodynamic drag of a new sports car is to be predicted at a speed of 50.0 mi/h at an air temperature of 25°C. Automotive engineers build a one-fifth scale model of the car to test in a wind tunnel. It is winter and the wind tunnel is located in an unheated building; the temperature of the wind tunnel air is only about 5°C. Determine how fast the engineers should run the wind tunnel in order to achieve similarity between the model and the prototype.

Properties For air at atmospheric pressure and at $T = 25°C$, $\rho = 1.184$ kg/m^3 and $\mu = 1.849 \times 10^{-5}$ kg/m·s. Similarly, at $T = 5°C$, $\rho = 1.269$ kg/m^3 and $\mu = 1.754 \times 10^{-5}$ kg/m·s.



A *drag balance* is a device used in a wind tunnel to measure the aerodynamic drag of a body. When testing automobile models, a *moving belt* is often added to the floor of the wind tunnel to simulate the moving ground (from the car's frame of reference).

Analysis Since there is only one independent Π in this problem, the similarity equation (Eq. 7–12) holds if $\Pi_{2,m} = \Pi_{2,p}$, where Π_2 is given by Eq. 7–13, and we call it the Reynolds number. Thus, we write

$$\Pi_{2,m} = \mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \Pi_{2,p} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

which we solve for the unknown wind tunnel speed for the model tests, V_m,

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

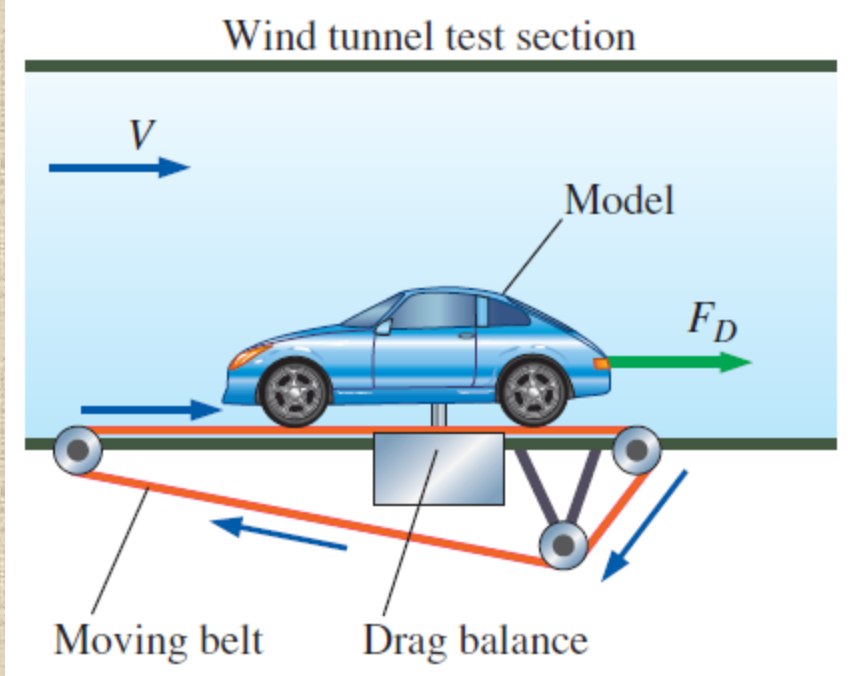
$$= (50.0\ \mathrm{mi/h})\left(\frac{1.754 \times 10^{-5}\ \mathrm{kg/m{\cdot}s}}{1.849 \times 10^{-5}\ \mathrm{kg/m{\cdot}s}}\right)\left(\frac{1.184\ \mathrm{kg/m^3}}{1.269\ \mathrm{kg/m^3}}\right)(5) = \mathbf{221\ mi/h}$$

Thus, to ensure similarity, the wind tunnel should be run at 221 mi/h (to three significant digits). Note that we were never given the actual length of either car, but the ratio of L_p to L_m is known because the prototype is five times larger than the scale model. When the dimensional parameters are rearranged as nondimensional ratios (as done here), the unit system is irrelevant. Since the units in each numerator cancel those in each denominator, no unit conversions are necessary.

Discussion This speed is quite high (about 100 m/s), and the wind tunnel may not be able to run at that speed. Furthermore, the incompressible approximation may come into question at this high speed (we discuss this in more detail in Example 7–8).

EXAMPLE 7–6 Prediction of Aerodynamic Drag Force on a Prototype Car

This example is a follow-up to Example 7–5. Suppose the engineers run the wind tunnel at 221 mi/h to achieve similarity between the model and the prototype. The aerodynamic drag force on the model car is measured with a **drag balance** (Fig. 7–19). Several drag readings are recorded, and the average drag force on the model is 21.2 lbf. Predict the aerodynamic drag force on the prototype (at 50 mi/h and 25°C).



A *drag balance* is a device used in a wind tunnel to measure the aerodynamic drag of a body. When testing automobile models, a *moving belt* is often added to the floor of the wind tunnel to simulate the moving ground (from the car's frame of reference).

Analysis The similarity equation (Eq. 7–12) shows that since $\Pi_{2,m} = \Pi_{2,p}$, $\Pi_{1,m} = \Pi_{1,p}$, where Π_1 is given for this problem by Eq. 7–13. Thus, we write

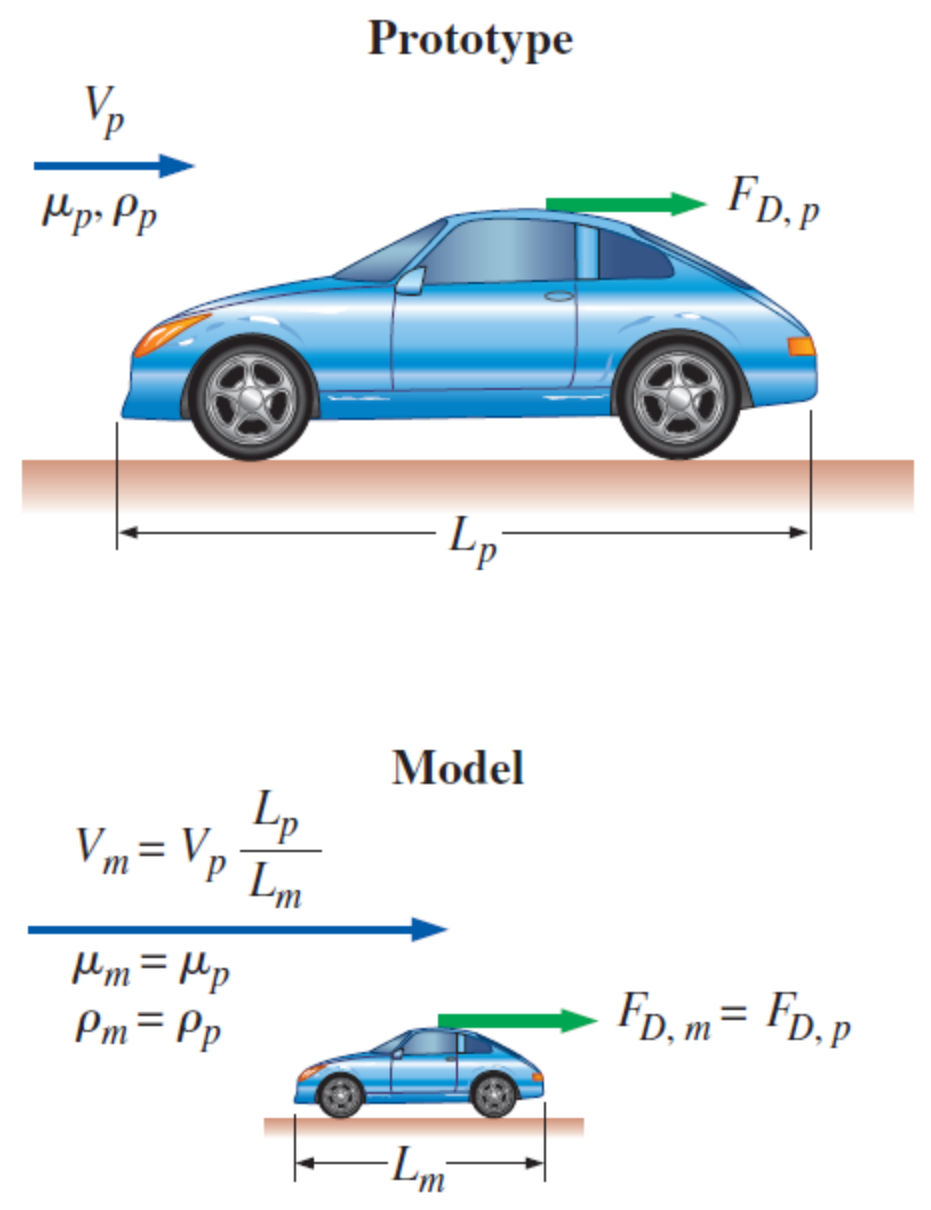
$$\Pi_{1,m} = \frac{F_{D,m}}{\rho_m V_m^2 L_m^2} = \Pi_{1,p} = \frac{F_{D,p}}{\rho_p V_p^2 L_p^2}$$

which we solve for the unknown aerodynamic drag force on the prototype car, $F_{D,p}$,

$$F_{D,p} = F_{D,m}\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

$$= (21.2\ \text{lbf})\left(\frac{1.184\ \text{kg/m}^3}{1.269\ \text{kg/m}^3}\right)\left(\frac{50.0\ \text{mi/h}}{221\ \text{mi/h}}\right)^2(5)^2 = \mathbf{25.3\ lbf}$$

Discussion By arranging the dimensional parameters as nondimensional ratios, the units cancel nicely even though they are a mixture of SI and English units. Because both velocity and length are squared in the equation for Π_1, the higher speed in the wind tunnel nearly compensates for the model's smaller size, and the drag force on the model is nearly the same as that on the prototype. In fact, if the density and viscosity of the air in the wind tunnel were *identical* to those of the air flowing over the prototype, the two drag forces would be identical as well (Fig. 7–20).



For the special case in which the wind tunnel air and the air flowing over the prototype have the same properties ($\rho_m = \rho_p$, $\mu_m = \mu_p$), and under similarity conditions ($V_m = V_p L_p / L_m$), the aerodynamic drag force on the prototype is equal to that on the scale model. If the two fluids do *not* have the same properties, the two drag forces are *not* necessarily the same, even under dynamically similar conditions.

If a water tunnel is used instead of a wind tunnel to test their one-fifth scale model, the water tunnel speed required to achieve similarity is

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0\ \text{mi/h})\left(\frac{1.002 \times 10^{-3}\ \text{kg/m} \cdot \text{s})}{1.849 \times 10^{-5}\ \text{kg/m} \cdot \text{s}}\right)\left(\frac{1.184\ \text{kg/m}^3}{998.0\ \text{kg/m}^3}\right)(5) = 16.1\ \text{mi/h}$$

One advantage of a water tunnel is that the required water tunnel speed is much lower than that required for a wind tunnel using the same size model (221 mi/h for air and 16.1 mi/h for water) .

Similarity can be achieved even when the model fluid is different than the prototype fluid. Here a submarine model is tested in a wind tunnel.

7–4 ■ THE METHOD OF REPEATING VARIABLES AND THE BUCKINGHAM PI THEOREM

How to *generate* the nondimensional parameters, i.e., the Π's?

There are several methods that have been developed for this purpose, but the most popular (and simplest) method is the **method of repeating variables**.

The Method of Repeating Variables

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,

$$k = n - j$$

Step 4: Choose j *repeating parameters*.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

A concise summary of the six steps that comprise the *method of repeating variables*.

Detailed description of the six steps that comprise the *method of repeating variables*[*]

Step 1 List the parameters (dimensional variables, nondimensional variables, and dimensional constants) and count them. Let n be the total number of parameters in the problem, including the dependent variable. Make sure that any listed independent parameter is indeed independent of the others, i.e., it cannot be expressed in terms of them. (E.g., don't include radius r and area $A = \pi r^2$, since r and A are *not* independent.)

Step 2 List the primary dimensions for each of the n parameters.

Step 3 Guess the **reduction** j. As a first guess, set j equal to the number of primary dimensions represented in the problem. The expected number of Π's (k) is equal to n minus j, according to the **Buckingham Pi theorem,**

The Buckingham Pi theorem: $$k = n - j \tag{7–14}$$

If at this step or during any subsequent step, the analysis does not work out, verify that you have included enough parameters in step 1. Otherwise, go back and *reduce j by one* and try again.

Step 4 Choose j **repeating parameters** that will be used to construct each Π. Since the repeating parameters have the potential to appear in each Π, be sure to choose them *wisely* (Table 7–3).

Step 5 Generate the Π's one at a time by grouping the j repeating parameters with one of the remaining parameters, forcing the product to be dimensionless. In this way, construct all k Π's. By convention the first Π, designated as Π_1, is the *dependent* Π (the one on the left side of the list). Manipulate the Π's as necessary to achieve established dimensionless groups (Table 7–5).

Step 6 Check that all the Π's are indeed dimensionless. Write the final functional relationship in the form of Eq. 7–11.

[*] This is a step-by-step method for finding the dimensionless Π groups when performing a dimensional analysis.

Step 1 *List of relevant parameters:* $z = f(t, w_0, z_0, g)$ $n = 5$

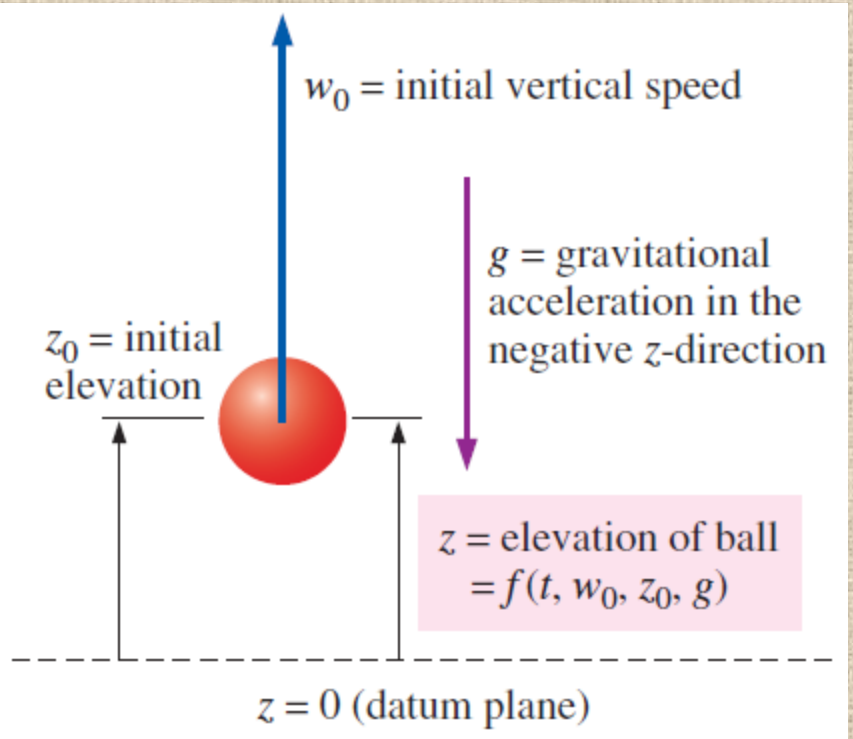
Step 2

z	t	w_0	z_0	g
$\{L^1\}$	$\{t^1\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{L^1t^{-2}\}$

Step 3 *Reduction:* $j = 2$

Number of expected Π*'s:* $k = n - j = 5 - 2 = 3$

Step 4 *Repeating parameters:* w_0 and z_0



Setup for dimensional analysis of a ball falling in a vacuum. Elevation z is a function of time t, initial vertical speed w_0, initial elevation z_0, and gravitational constant g.

Step 5

Dependent Π: $\Pi_1 = z w_0^{a_1} z_0^{b_1}$

Dimensions of Π_1: $\{\Pi_1\} = \{L^0 t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1 (L^1 t^{-1})^{a_1} L^{b_1}\}$

Time: $\{t^0\} = \{t^{-a_1}\}$ $\quad 0 = -a_1 \quad a_1 = 0$

Length: $\{L^0\} = \{L^1 L^{a_1} L^{b_1}\}$ $\quad 0 = 1 + a_1 + b_1 \quad b_1 = -1 - a_1 \quad b_1 = -1$

$$\Pi_1 = \frac{z}{z_0}$$

First independent Π: $\Pi_2 = t w_0^{a_2} z_0^{b_2}$

Dimensions of Π_2: $\{\Pi_2\} = \{L^0 t^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{t (L^1 t^{-1})^{a_2} L^{b_2}\}$

Equating exponents,

Time: $\{t^0\} = \{t^1 t^{-a_2}\}$ $\quad 0 = 1 - a_2 \quad a_2 = 1$

Length: $\{L^0\} = \{L^{a_2} L^{b_2}\}$ $\quad 0 = a_2 + b_2 \quad b_2 = -a_2 \quad b_2 = -1$

$$\Pi_2 = \frac{w_0 t}{z_0}$$

Second independent Π: $\Pi_3 = g w_0^{a_3} z_0^{b_3}$

Dimensions of Π_3: $\{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$

Equating exponents,

Time: $\{t^0\} = \{t^{-2}t^{-a_3}\}$ $\quad 0 = -2 - a_3 \quad a_3 = -2$

Length: $\{L^0\} = \{L^1L^{a_3}L^{b_3}\}$ $\quad 0 = 1 + a_3 + b_3 \quad b_3 = -1 - a_3 \quad b_3 = 1$

$$\Pi_3 = \frac{gz_0}{w_0^2}$$

Modified Π_3:

$$\Pi_{3,\text{ modified}} = \left(\frac{gz_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{gz_0}} = \text{Fr}$$

Multiplication: Add exponents

$$x^a x^b x^{2c} = x^{a+b+2c}$$

Division: Subtract exponents

$$\frac{x^a}{x^b} \times \frac{1}{x^{2c}} = x^{a-b-2c}$$

The mathematical rules for adding and subtracting exponents during multiplication and division, respectively.

Hint of the Day

A wise choice of repeating parameters for most fluid flow problems is a length, a velocity, and a mass or density.

It is wise to choose *common* parameters as repeating parameters since they may appear in each of your dimensionless Π groups.

$$\{\Pi_1\} = \{m^0L^0t^0T^0I^0C^0N^0\} = \{1\}$$

$$\{\Pi_2\} = \{m^0L^0t^0T^0I^0C^0N^0\} = \{1\}$$

$$\vdots$$

$$\{\Pi_k\} = \{m^0L^0t^0T^0I^0C^0N^0\} = \{1\}$$

The Π groups that result from the method of repeating variables are guaranteed to be dimensionless because we *force* the overall exponent of all seven primary dimensions to be zero.

Established nondimensional parameters are usually named after a notable scientist or engineer.

Step 6

Relationship between Π*'s:* $\quad \Pi_1 = f(\Pi_2, \Pi_3) \quad \rightarrow \quad \dfrac{z}{z_0} = f\left(\dfrac{w_0 t}{z_0}, \dfrac{w_0}{\sqrt{g z_0}}\right)$

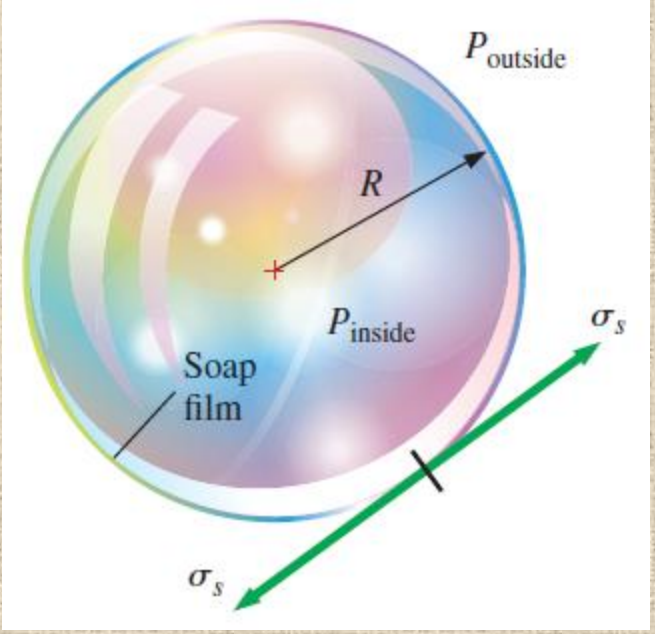
Final result of dimensional analysis: $\quad z^* = f(t^*, \mathrm{Fr})$

The method of repeating variables cannot predict the exact mathematical form of the equation.

A quick check of your algebra is always wise.

EXAMPLE 7–7 Pressure in a Soap Bubble

Some children are playing with soap bubbles, and you become curious as to the relationship between soap bubble radius and the pressure inside the soap bubble (Fig. 7–29). You reason that the pressure inside the soap bubble must be greater than atmospheric pressure, and that the shell of the soap bubble is under tension, much like the skin of a balloon. You also know that the property surface tension must be important in this problem. Not knowing any other physics, you decide to approach the problem using dimensional analysis. Establish a relationship between pressure difference $\Delta P = P_{\text{inside}} - P_{\text{outside}}$, soap bubble radius R, and the surface tension σ_s of the soap film.



The pressure inside a soap bubble is greater than that surrounding the soap bubble due to surface tension in the soap film.

Analysis The step-by-step method of repeating variables is employed.

Step 1 There are three variables and constants in this problem; $n = 3$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $\Delta P = f(R, \sigma_s) \quad n = 3$

Step 2 The primary dimensions of each parameter are listed. The dimensions of surface tension are obtained from Example 7–1, and those of pressure from Example 7–2.

ΔP	R	σ_s
$\{m^1L^{-1}t^{-2}\}$	$\{L^1\}$	$\{m^1t^{-2}\}$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction (first guess): $j = 3$

If this value of j is correct, the expected number of Π's is $k = n - j = 3 - 3 = 0$. But how can we have zero Π's? Something is obviously not right (Fig. 7–30). At times like this, we need to first go back and make sure that we are not neglecting some important variable or constant in the problem. Since we are confident that the pressure difference should depend only on soap bubble radius and surface tension, we reduce the value of j by one,

Reduction (second guess): $j = 2$

If this value of j is correct, $k = n - j = 3 - 2 = 1$. Thus we expect *one* Π, which is more physically realistic than zero Π's.

What happens if

$$k = n - j = 0?$$

Do the following:

- **Check your list of parameters.**
- **Check your algebra.**
- **If all else fails, reduce j by one.**

If the method of repeating variables indicates zero Π's, we have either made an error, or we need to reduce j by one and start over.

Step 4 We need to choose two repeating parameters since $j = 2$. Following the guidelines of Table 7–3, our only choices are R and σ_s, since ΔP is the dependent variable.

Step 5 We combine these repeating parameters into a product with the dependent variable ΔP to create the dependent Π,

Dependent Π:
$$\Pi_1 = \Delta P R^{a_1} \sigma_s^{b_1} \tag{1}$$

We apply the primary dimensions of step 2 into Eq. 1 and force the Π to be dimensionless.

Dimensions of Π_1:
$$\{\Pi_1\} = \{m^0 L^0 t^0\} = \{\Delta P R^{a_1} \sigma_s^{b_1}\} = \{(m^1 L^{-1} t^{-2}) L^{a_1} (m^1 t^{-2})^{b_1}\}$$

We equate the exponents of each primary dimension to solve for a_1 and b_1:

Time:	$\{t^0\} = \{t^{-2} t^{-2b_1}\}$	$0 = -2 - 2b_1$	$b_1 = -1$
Mass:	$\{m^0\} = \{m^1 m^{b_1}\}$	$0 = 1 + b_1$	$b_1 = -1$
Length:	$\{L^0\} = \{L^{-1} L^{a_1}\}$	$0 = -1 + a_1$	$a_1 = 1$

Fortunately, the first two results agree with each other, and Eq. 1 thus becomes

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} \tag{2}$$

From Table 7–5, the established nondimensional parameter most similar to Eq. 2 is the **Weber number**, defined as a pressure (ρV^2) times a length divided by surface tension. There is no need to further manipulate this Π.

Step 6 We write the final functional relationship. In the case at hand, there is only one Π, which is a function of *nothing*. This is possible only if the Π is constant. Putting Eq. 2 into the functional form of Eq. 7–11,

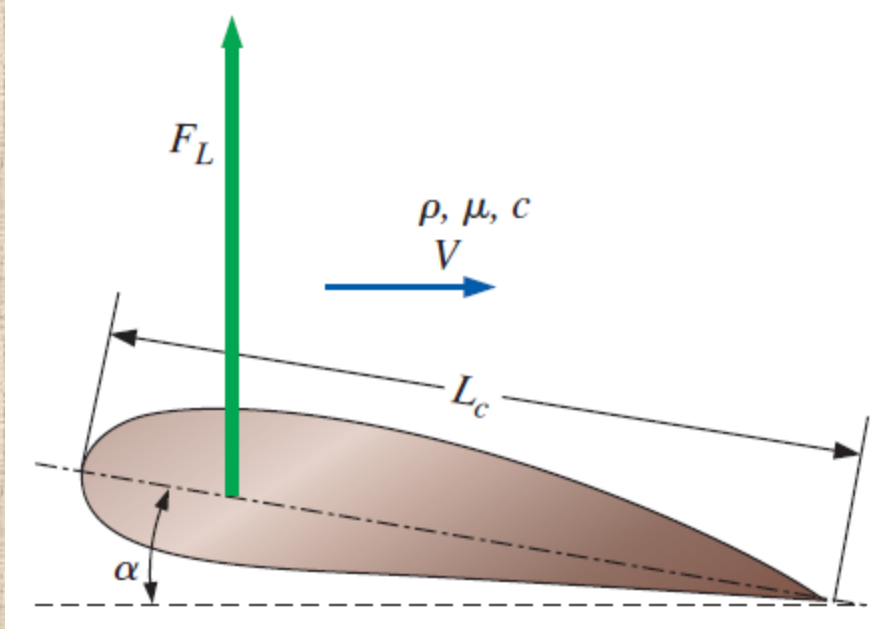
Relationship between Π*'s*:

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} = f(\text{nothing}) = \text{constant} \quad \rightarrow \quad \Delta P = \text{constant}\,\frac{\sigma_s}{R} \tag{3}$$

Discussion This is an example of how we can sometimes predict *trends* with dimensional analysis, even without knowing much of the physics of the problem. For example, we know from our result that if the radius of the soap bubble doubles, the pressure difference decreases by a factor of 2. Similarly, if the value of surface tension doubles, ΔP increases by a factor of 2. Dimensional analysis cannot predict the value of the constant in Eq. 3; further analysis (or *one* experiment) reveals that the constant is equal to 4 (Chap. 2).

EXAMPLE 7–8 Lift on a Wing

Some aeronautical engineers are designing an airplane and wish to predict the lift produced by their new wing design (Fig. 7–31). The chord length L_c of the wing is 1.12 m, and its **planform area** A (area viewed from the top when the wing is at zero angle of attack) is 10.7 m^2. The prototype is to fly at V = 52.0 m/s close to the ground where T = 25°C. They build a one-tenth scale model of the wing to test in a pressurized wind tunnel. The wind tunnel can be pressurized to a maximum of 5 atm. At what speed and pressure should they run the wind tunnel in order to achieve dynamic similarity?



Lift F_L on a wing of chord length L_c at angle of attack α in a flow of free-stream speed V with density ρ, viscosity μ, and speed of sound c. The angle of attack α is measured relative to the free-stream flow direction.

Analysis First, the step-by-step method of repeating variables is employed to obtain the nondimensional parameters. Then, the dependent Π's are matched between prototype and model.

Step 1 There are seven parameters (variables and constants) in this problem; $n = 7$. They are listed in functional form, with the dependent variable listed as a function of the independent parameters:

List of relevant parameters: $$F_L = f(V, L_c, \rho, \mu, c, \alpha) \qquad n = 7$$

where F_L is the lift force on the wing, V is the fluid speed, L_c is the chord length, ρ is the fluid density, μ is the fluid viscosity, c is the speed of sound in the fluid, and α is the angle of attack of the wing.

Step 2 The primary dimensions of each parameter are listed; angle α is dimensionless:

F_L	V	L_c	ρ	μ	c	α
$\{m^1L^1t^{-2}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1t^{-1}\}$	$\{1\}$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction: $$j = 3$$

If this value of j is correct, the expected number of Π's is $k = n - j = 7 - 3 = 4$.

Step 4 We need to choose three repeating parameters since $j = 3$. Following the guidelines listed in Table 7–3, we cannot pick the dependent variable F_L. Nor can we pick α since it is already dimensionless. We cannot choose both V and c since their dimensions are identical. It would not be desirable to have μ appear in all the Π's. The best choice of repeating parameters is thus either V, L_c, and ρ or c, L_c, and ρ. Of these, the former is the better choice since the speed of sound appears in only one of the established nondimensional parameters of Table 7–5, whereas the velocity scale is more "common" and appears in several of the parameters (Fig. 7–32).

Repeating parameters: V, L_c, and ρ

Step 5 The dependent Π is generated:

$$\Pi_1 = F_L V^{a_1} L_c^{b_1} \rho^{c_1} \quad \rightarrow \quad \{\Pi_1\} = \{(\mathrm{m}^1\mathrm{L}^1\mathrm{t}^{-2})(\mathrm{L}^1\mathrm{t}^{-1})^{a_1}(\mathrm{L}^1)^{b_1}(\mathrm{m}^1\mathrm{L}^{-3})^{c_1}\}$$

The exponents are calculated by forcing the Π to be dimensionless (algebra not shown). We get $\boldsymbol{a_1 = -2}$, $\boldsymbol{b_1 = -2}$, and $\boldsymbol{c_1 = -1}$. The dependent Π is thus

$$\Pi_1 = \frac{F_L}{\rho V^2 L_c^2}$$

From Table 7–5, the established nondimensional parameter most similar to our Π_1 is the **lift coefficient**, defined in terms of planform area A rather than the square of chord length, and with a factor of 1/2 in the denominator. Thus, we may manipulate this Π according to the guidelines listed in Table 7–4 as follows:

Modified Π_1:

$$\Pi_{1,\,\text{modified}} = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \text{Lift coefficient} = C_L$$

Oftentimes when performing the method of repeating variables, the most difficult part of the procedure is choosing the repeating parameters. With practice, however, you will learn to choose these parameters wisely.

Similarly, the first independent Π is generated:

$$\Pi_2 = \mu V^{a_2} L_c^{b_2} \rho^{c_2} \quad \rightarrow \quad \{\Pi_2\} = \{(\mathrm{m}^1\mathrm{L}^{-1}\mathrm{t}^{-1})(\mathrm{L}^1\mathrm{t}^{-1})^{a_2}(\mathrm{L}^1)^{b_2}(\mathrm{m}^1\mathrm{L}^{-3})^{c_2}\}$$

from which $a_2 = -1$, $b_2 = -1$, and $c_2 = -1$, and thus

$$\Pi_2 = \frac{\mu}{\rho V L_c}$$

We recognize this Π as the inverse of the Reynolds number. So, after inverting,

Modified Π_2:

$$\Pi_{2,\,\text{modified}} = \frac{\rho V L_c}{\mu} = \text{Reynolds number} = \text{Re}$$

The third Π is formed with the speed of sound, the details of which are left for you to generate on your own. The result is

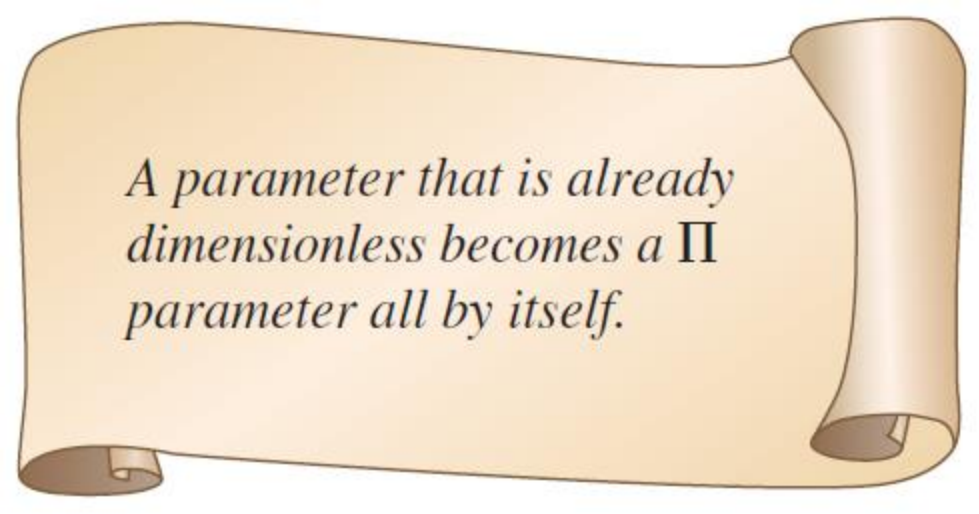
$$\Pi_3 = \frac{V}{c} = \text{Mach number} = \text{Ma}$$

Finally, since the angle of attack α is already dimensionless, it is a dimensionless Π group all by itself (Fig. 7–33). You are invited to go through the algebra; you will find that all the exponents turn out to be zero, and thus

$$\Pi_4 = \alpha = \text{Angle of attack}$$

Step 6 We write the final functional relationship as

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = f(\mathrm{Re}, \mathrm{Ma}, \alpha) \qquad (1)$$



A parameter that is dimensionless (like an angle) is already a nondimensional Π all by itself—we know this Π without doing any further algebra.

To achieve dynamic similarity, Eq. 7–12 requires that all three of the dependent nondimensional parameters in Eq. 1 match between the model and the prototype. While it is trivial to match the angle of attack, it is not so simple to simultaneously match the Reynolds number and the Mach number. For example, if the wind tunnel were run at the same temperature and pressure as those of the prototype, such that ρ, μ, and c of the air flowing over the model were the same as ρ, μ, and c of the air flowing over the prototype, Reynolds number similarity would be achieved by setting the wind tunnel air speed to 10 times that of the prototype (since the model is one-tenth scale). But then the Mach numbers would differ by a factor of 10. At 25°C, c is approximately 346 m/s, and the Mach number of the prototype airplane wing is $Ma_p = 52.0/346 = 0.150$—subsonic. At the required wind tunnel speed, Ma_m would be 1.50—supersonic! This is clearly unacceptable since the physics of the flow changes dramatically from subsonic to supersonic conditions. At the other extreme, if we were to match Mach numbers, the Reynolds number of the model would be 10 times too small.

What should we do? A common rule of thumb is that for Mach numbers less than about 0.3, as is the fortunate case here, compressibility effects are practically negligible. Thus, it is not necessary to exactly match the Mach number; rather, as long as Ma_m is kept below about 0.3, approximate dynamic similarity can be achieved by matching the Reynolds number. Now the problem shifts to one of how to match Re while maintaining a low Mach number. This is where the pressurization feature of the wind tunnel comes in. At constant temperature, density is proportional to pressure, while viscosity and speed of sound are very weak functions of pressure. If the wind tunnel pressure could be pumped to 10 atm, we could run the model test at the

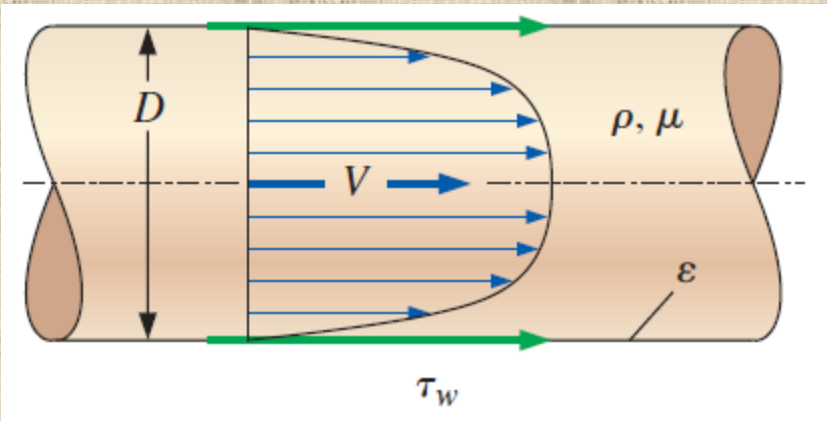
same speed as the prototype and achieve a nearly perfect match in both Re and Ma. However, at the maximum wind tunnel pressure of 5 atm, the required wind tunnel speed would be twice that of the prototype, or 104 m/s. The Mach number of the wind tunnel model would thus be Ma_m = 104/346 = 0.301—approximately at the limit of incompressibility according to our rule of thumb. In summary, the wind tunnel should be run at approximately **100 m/s**, **5 atm**, and **25°C**.

Discussion This example illustrates one of the (frustrating) limitations of dimensional analysis; namely, *You may not always be able to match all the dependent* Π*'s simultaneously in a model test*. Compromises must be made in which only the most important Π's are matched. In many practical situations in fluid mechanics, the Reynolds number is not critical for dynamic similarity, provided that Re is high enough. If the Mach number of the prototype were significantly larger than about 0.3, we would be wise to precisely match the Mach number rather than the Reynolds number in order to ensure reasonable results. Furthermore, if a different gas were used to test the model, we would also need to match the specific heat ratio (k), since compressible flow behavior is strongly dependent on k (Chap. 12). We discuss such model testing problems in more detail in Section 7–5.

In Examples 7–5 and 7–6 the air speed of the prototype car is 50.0 mi/h, and that of the wind tunnel is 224 mi/h. At 25°C, this corresponds to a prototype Mach number of Ma_p = 0.065, and at 5°C, the Mach number of the wind tunnel is 0.29—on the borderline of the incompressible limit. In hindsight, we should have included the speed of sound in our dimensional analysis, which would have generated the Mach number as an additional . Another way to match the Reynolds number while keeping the Mach number low is to use a *liquid* such as water, since liquids are nearly ncompressible, even at fairly high speeds.

EXAMPLE 7–9 Friction in a Pipe

Consider flow of an incompressible fluid of density ρ and viscosity μ through a long, horizontal section of round pipe of diameter D. The velocity profile is sketched in Fig. 7–34; V is the average speed across the pipe cross section, which by conservation of mass remains constant down the pipe. For a very long pipe, the flow eventually becomes hydrodynamically **fully developed,** which means that the velocity profile also remains uniform down the pipe. Because of frictional forces between the fluid and the pipe wall, there exists a shear stress τ_w on the inside pipe wall as sketched. The shear stress is also constant down the pipe in the fully developed region. We assume some constant average roughness height ε along the inside wall of the pipe. In fact, the only parameter that is *not* constant down the length of pipe is the pressure, which must decrease (linearly) down the pipe in order to "push" the fluid through the pipe to overcome friction. Develop a nondimensional relationship between shear stress τ_w and the other parameters in the problem.



Friction on the inside wall of a pipe. The shear stress τ_w on the pipe walls is a function of average fluid speed V, average wall roughness height ε, fluid density ρ, fluid viscosity μ, and inside pipe diameter D.

Analysis The step-by-step method of repeating variables is employed to obtain the nondimensional parameters.

Step 1 There are six variables and constants in this problem; $n = 6$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $\tau_w = f(V, \varepsilon, \rho, \mu, D) \quad n = 6$

Step 2 The primary dimensions of each parameter are listed. Note that shear stress is a force per unit area, and thus has the same dimensions as pressure.

τ_w	V	ε	ρ	μ	D
$\{m^1L^{-1}t^{-2}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1\}$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction: $j = 3$

If this value of j is correct, the expected number of Π's is $k = n - j = 6 - 3 = 3$.

Step 4 We choose three repeating parameters since $j = 3$. Following the guidelines of Table 7–3, we cannot pick the dependent variable τ_w. We cannot choose both ε and D since their dimensions are identical, and it would not be desirable to have μ or ε appear in all the Π's. The best choice of repeating parameters is thus V, D, and ρ.

Repeating parameters: V, D, and ρ

Step 5 The dependent Π is generated:

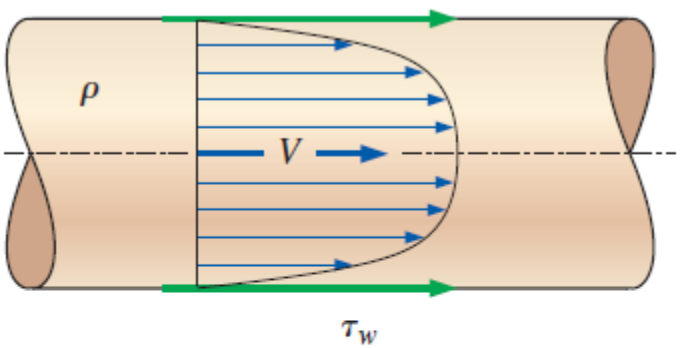
$$\Pi_1 = \tau_w V^{a_1} D^{b_1} \rho^{c_1} \quad \rightarrow \quad \{\Pi_1\} = \{(m^1L^{-1}t^{-2})(L^1t^{-1})^{a_1}(L^1)^{b_1}(m^1L^{-3})^{c_1}\}$$

from which $a_1 = -2$, $b_1 = 0$, and $c_1 = -1$, and thus the dependent Π is

$$\Pi_1 = \frac{\tau_w}{\rho V^2}$$

From Table 7–5, the established nondimensional parameter most similar to this Π_1 is the **Darcy friction factor**, defined with a factor of 8 in the numerator (Fig. 7–35). Thus, we manipulate this Π according to the guidelines listed in Table 7–4 as follows:

Modified Π_1: $$\Pi_{1,\,\text{modified}} = \frac{8\tau_w}{\rho V^2} = \text{Darcy friction factor} = f$$



Darcy friction factor: $f = \dfrac{8\tau_w}{\rho V^2}$

Fanning friction factor: $C_f = \dfrac{2\tau_w}{\rho V^2}$

Although the *Darcy friction factor* for pipe flows is most common, you should be aware of an alternative, less common friction factor called the *Fanning friction factor*. The relationship between the two is $f = 4C_f$.

Similarly, the two independent Π's are generated, the details of which are left for you to do on your own:

$$\Pi_2 = \mu V^{a_2} D^{b_2} \rho^{c_2} \quad \rightarrow \quad \Pi_2 = \frac{\rho V D}{\mu} = \text{Reynolds number} = \text{Re}$$

$$\Pi_3 = \varepsilon V^{a_3} D^{b_3} \rho^{c_3} \quad \rightarrow \quad \Pi_3 = \frac{\varepsilon}{D} = \text{Roughness ratio}$$

Step 6 We write the final functional relationship as

$$f = \frac{8\tau_w}{\rho V^2} = f\left(Re, \frac{\varepsilon}{D}\right) \tag{1}$$

Discussion The result applies to both laminar and turbulent fully developed pipe flow; it turns out, however, that the second independent Π (roughness ratio ε/D) is not nearly as important in laminar pipe flow as in turbulent pipe flow. This problem presents an interesting connection between geometric similarity and dimensional analysis. Namely, it is necessary to match ε/D since it is an independent Π in the problem. From a different perspective, thinking of roughness as a geometric property, it is necessary to match ε/D to ensure *geometric similarity* between two pipes.

To verify the validity of Eq. 1 of Example 7–9, we use **computational fluid dynamics (CFD)** to predict the velocity profiles and the values of wall shear stress for two physically different but dynamically similar pipe flows:

- ***Air*** at 300 K flowing at an average speed of 14.5 ft/s through a pipe of inner diameter 1.00 ft and average roughness height 0.0010 ft.
- ***Water*** at 300 K flowing at an average speed of 3.09 m/s through a pipe of inner diameter 0.0300 m and average roughness height 0.030 mm.

The two pipes are clearly geometrically similar since they are both round pipes.

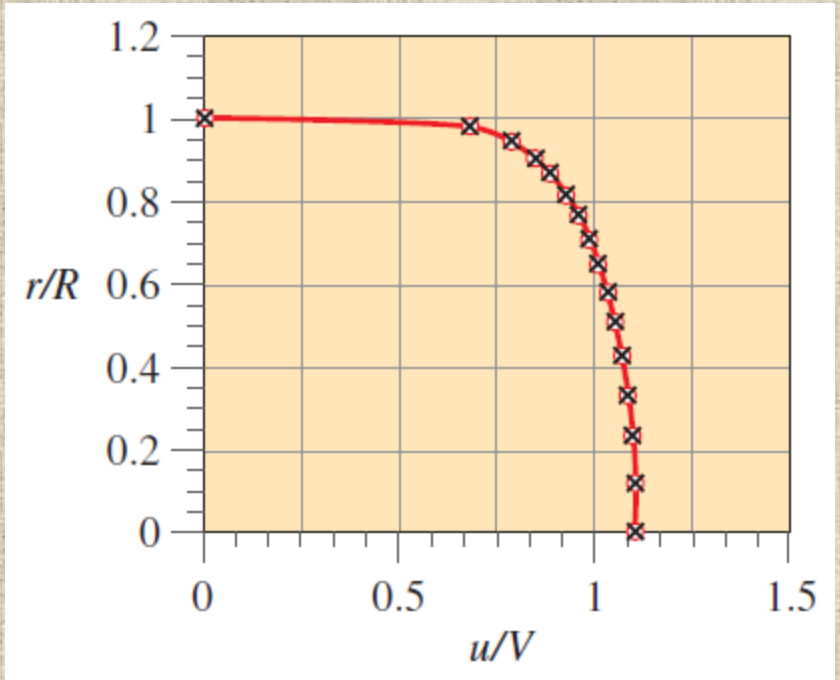
They have the same average roughness ratio (ε/D = 0.0010 in both cases).

We have carefully chosen the values of average speed and diameter such that the two flows are also *dynamically* similar.

Specifically, the other independent Π (the Reynolds number) also matches between the two flows.

$$\mathrm{Re}_{\mathrm{air}} = \frac{\rho_{\mathrm{air}} V_{\mathrm{air}} D_{\mathrm{air}}}{\mu_{\mathrm{air}}} = \frac{(1.225\ \mathrm{kg/m^3})(14.5\ \mathrm{ft/s})(1.00\ \mathrm{ft})}{1.789 \times 10^{-5}\ \mathrm{kg/m \cdot s}} \left(\frac{0.3048\ \mathrm{m}}{\mathrm{ft}}\right)^2 = 9.22 \times 10^4$$

$$\mathrm{Re}_{\mathrm{water}} = \frac{\rho_{\mathrm{water}} V_{\mathrm{water}} D_{\mathrm{water}}}{\mu_{\mathrm{water}}} = \frac{(998.2\ \mathrm{kg/m^3})(3.09\ \mathrm{m/s})(0.0300\ \mathrm{m})}{0.001003\ \mathrm{kg/m \cdot s}} = 9.22 \times 10^4$$



Normalized axial velocity profiles for fully developed flow through a pipe as predicted by CFD; profiles of air (circles) and water (crosses) are shown on the same plot.

Comparison of wall shear stress and nondimensionalized wall shear stress for fully developed flow through an air pipe and a water pipe as predicted by CFD*

Parameter	Air Flow	Water Flow
Wall shear stress	$\tau_{w,\,\text{air}} = 0.0557\ \text{N/m}^2$	$\tau_{w,\,\text{water}} = 22.2\ \text{N/m}^2$
Dimensionless wall shear stress (Darcy friction factor)	$f_{\text{air}} = \dfrac{8\tau_{w,\,\text{air}}}{\rho_{\text{air}} V_{\text{air}}^2} = 0.0186$	$f_{\text{water}} = \dfrac{8\tau_{w,\,\text{water}}}{\rho_{\text{water}} V_{\text{water}}^2} = 0.0186$

* Data obtained with ANSYS-FLUENT using the standard k-ε turbulence model with wall functions.

7–5 ■ EXPERIMENTAL TESTING, MODELING, AND INCOMPLETE SIMILARITY

One of the most useful applications of dimensional analysis is in designing physical and/or numerical experiments, and in reporting the results of such experiments.

In this section we discuss both of these applications, and point out situations in which complete dynamic similarity is not achievable.

Setup of an Experiment and Correlation of Experimental Data

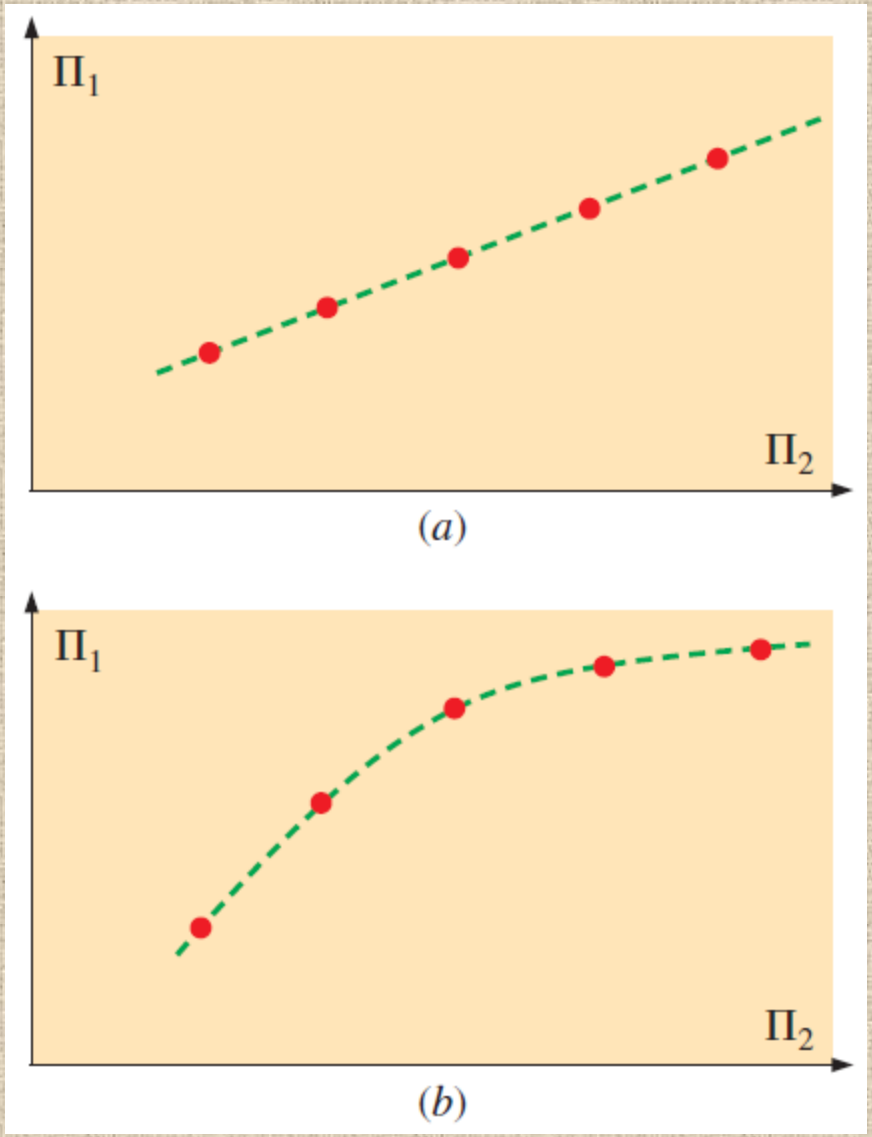
Consider a problem in which there are five original parameters (one of which is the *dependent* parameter).

A complete set of experiments (called a **full factorial** test matrix) is conducted.

This testing would require $5^4 = 625$ experiments.

Assuming that three primary dimensions are represented in the problem, we can reduce the number of parameters from five to two ($k = 5 - 3 = 2$ nondimensional groups), and the number of *independent* parameters from four to one.

Thus, for the same resolution we would then need to conduct a total of only $5^1 = 5$ experiments.



For a two-Π problem, we plot dependent dimensionless parameter (Π_1) as a function of independent dimensionless parameter (Π_2). The resulting plot can be (*a*) linear or (*b*) nonlinear. In either case, regression and curve-fitting techniques are available to determine the relationship between the Π's.

If there are more than two Π's in the problem (e.g., a three- Π problem ora four- Π problem), we need to set up a test matrix to determine the relationship between the dependent Π and the independent Π's. In many cases we discover that one or more of the dependent Π's has negligible effect and can be removed from the list of necessary dimensionless parameters.

Incomplete Similarity

We have shown several examples in which the nondimensional groups are easily obtained with paper and pencil through straightforward use of the method of repeating variables.

In fact, after sufficient practice, you should be able to obtain the Π's with ease—sometimes in your head or on the "back of an envelope."

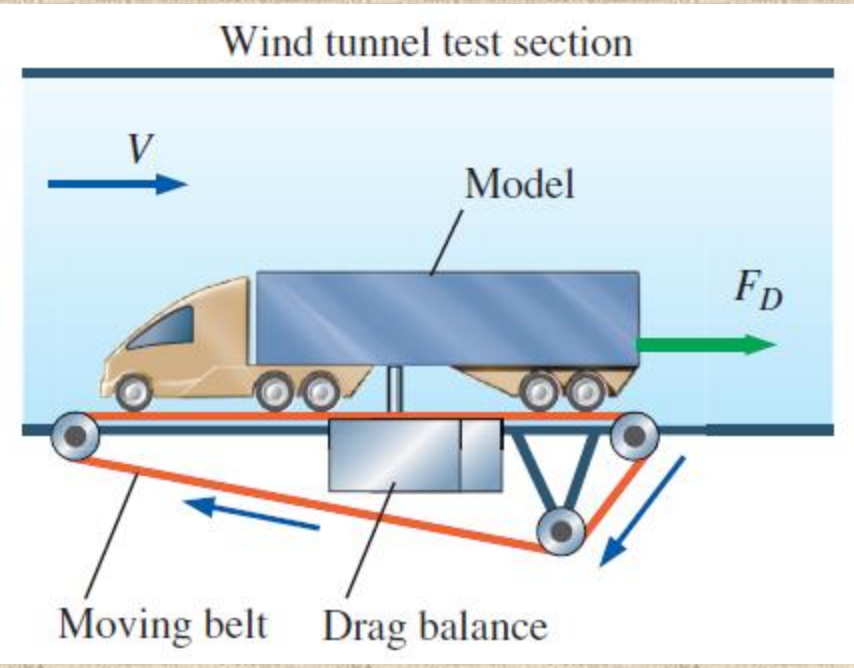
Unfortunately, it is often a much different story when we go to apply the results of our dimensional analysis to experimental data.

The problem is that it is not always possible to match *all* the Π's of a model to the corresponding 's of the prototype, even if we are careful to achieve geometric similarity.

This situation is called **incomplete similarity**.

Fortunately, in some cases of incomplete similarity, we are still able to extrapolate model test data to obtain reasonable full-scale predictions.

Wind Tunnel Testing



Measurement of aerodynamic drag on a model truck in a wind tunnel equipped with a *drag balance* and a *moving belt* ground plane.

We illustrate incomplete similarity with the problem of measuring the aerodynamic drag force on a model truck in a wind tunnel.

One-sixteenth scale.

The model is geometrically similar to the prototype.

The model truck is 0.991 m long. Wind tunnel has a maximum speed of 70 m/s.

The wind tunnel test section is 1.0 m tall and 1.2 m wide.

$$\mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right) = (26.8\ \mathrm{m/s})(1)(1)\left(\frac{16}{1}\right) = 429\ \mathrm{m/s}$$

To match the Reynolds number between model and prototype, the wind tunnel should be run at 429 m/s. This is impossible in this wind tunnel.

What do we do? There are several options:

(1) Use a bigger wind tunnel. Automobile manufacturers typically test with three-eighths scale model cars and with one-eighth scale model trucks and buses in very large wind tunnels.

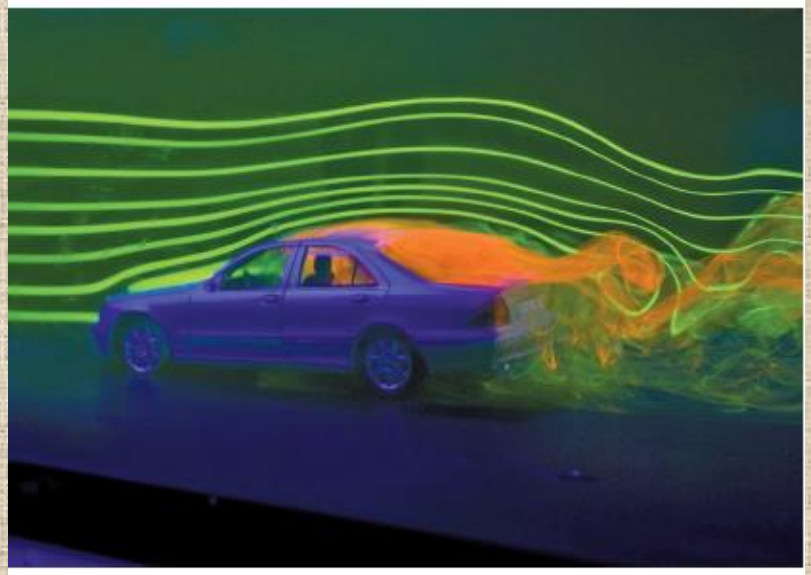
(2) We could use a different fluid for the model tests. For example, water can achieve higher Re numbers, but more expensive.

(3) We could pressurize the wind tunnel and/or adjust the air temperature to increase the maximum Reynolds number capability.

(4) If all else fails, we could run the wind tunnel at several speeds near the maximum speed, and then extrapolate our results to the full-scale Reynolds number.

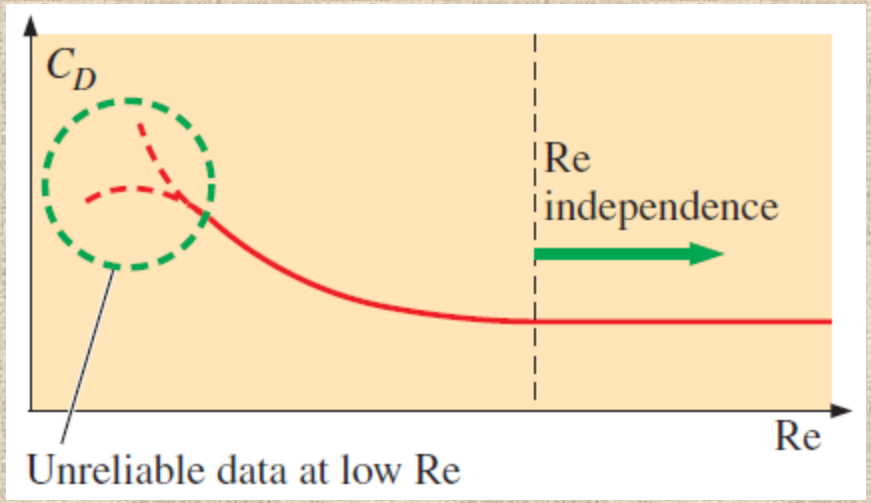
Fortunately, it turns out that for many wind tunnel tests the last option is quite viable.

(*a*)



(*b*)

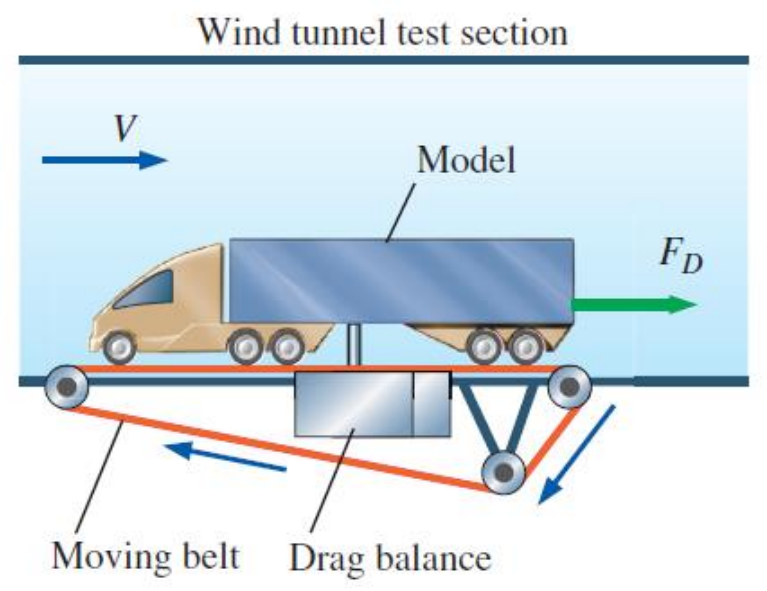
(*a*) *The Langley full-scale wind tunnel* (LFST) is large enough that full-scale vehicles can be tested. (*b*) *For the* same scale model and speed, water tunnels achieve higher Reynolds numbers than wind tunnels.



For many objects, the drag coefficient levels off at Reynolds numbers above some threshold value. This fortunate situation is called *Reynolds number independence*. It enables us to extrapolate to prototype Reynolds numbers that are outside of the range of our experimental facility.

EXAMPLE 7–10 Model Truck Wind Tunnel Measurements

A one-sixteenth scale model tractor-trailer truck (18-wheeler) is tested in a wind tunnel as sketched in Fig. 7–38. The model truck is 0.991 m long, 0.257 m tall, and 0.159 m wide. During the tests, the moving ground belt speed is adjusted so as to always match the speed of the air moving through the test section. Aerodynamic drag force F_D is measured as a function of wind tunnel speed; the experimental results are listed in Table 7–7. Plot the drag coefficient C_D as a function of the Reynolds number Re, where the area used for the calculation of C_D is the frontal area of the model truck (the area you see when you look at the model from upstream), and the length scale used for calculation of Re is truck width W. Have we achieved dynamic similarity? Have we achieved Reynolds number independence in our wind tunnel test? Estimate the aerodynamic drag force on the prototype truck traveling on the highway at 26.8 m/s. Assume that both the wind tunnel air and the air flowing over the prototype car are at 25°C and standard atmospheric pressure.



Measurement of aerodynamic drag on a model truck in a wind tunnel equipped with a *drag balance* and a *moving belt* ground plane.

Properties For air at atmospheric pressure and at $T = 25°C$, $\rho = 1.184\ kg/m^3$ and $\mu = 1.849 \times 10^{-5}\ kg/m{\cdot}s$.

Analysis We calculate C_D and Re for the last data point listed in Table 7–7 (at the fastest wind tunnel speed),

$$C_{D,m} = \frac{F_{D,m}}{\frac{1}{2}\rho_m V_m^2 A_m} = \frac{89.9\ \text{N}}{\frac{1}{2}(1.184\ \text{kg/m}^3)(70\ \text{m/s})^2(0.159\ \text{m})(0.257\ \text{m})}\left(\frac{1\ \text{kg}\cdot\text{m/s}^2}{1\ \text{N}}\right)$$

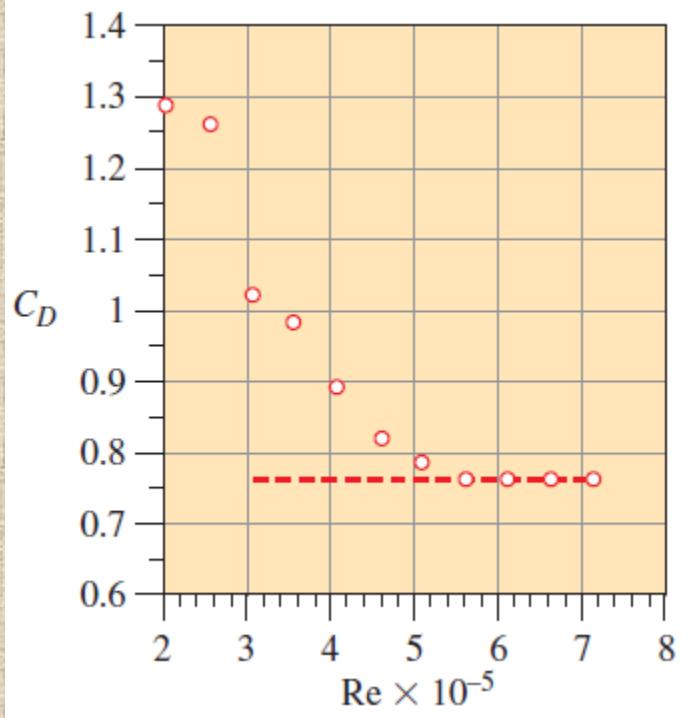
$$= 0.758$$

and

$$\text{Re}_m = \frac{\rho_m V_m W_m}{\mu_m} = \frac{(1.184\ \text{kg/m}^3)(70\ \text{m/s})(0.159\ \text{m})}{1.849 \times 10^{-5}\ \text{kg/m}\cdot\text{s}} = 7.13 \times 10^5 \qquad (1)$$

Wind tunnel data: aerodynamic drag force on a model truck as a function of wind tunnel speed

V, m/s	F_D, N
20	12.4
25	19.0
30	22.1
35	29.0
40	34.3
45	39.9
50	47.2
55	55.5
60	66.0
65	77.6
70	89.9



Aerodynamic drag coefficient as a function of the Reynolds number. The values are calculated from wind tunnel test data on a model truck (Table 7–7).

We repeat these calculations for all the data points in Table 7–7, and we plot C_D versus Re in Fig. 7–41.

Have we achieved dynamic similarity? Well, we have *geometric* similarity between model and prototype, but the Reynolds number of the prototype truck is

$$\mathrm{Re}_p = \frac{\rho_p V_p W_p}{\mu_p} = \frac{(1.184\ \mathrm{kg/m^3})(26.8\ \mathrm{m/s})[16(0.159\ \mathrm{m})]}{1.849 \times 10^{-5}\ \mathrm{kg/m{\cdot}s}} = 4.37 \times 10^6 \qquad \textbf{(2)}$$

where the width of the prototype is specified as 16 times that of the model. Comparison of Eqs. 1 and 2 reveals that the prototype Reynolds number is more than six times larger than that of the model. Since we cannot match the independent Π's in the problem, **dynamic similarity has not been achieved**.

Have we achieved Reynolds number independence? From Fig. 7–41 we see that **Reynolds number independence has indeed been achieved**—at Re greater than about 5×10^5, C_D has leveled off to a value of about 0.76 (to two significant digits).

Since we have achieved Reynolds number independence, we can extrapolate to the full-scale prototype, assuming that C_D remains constant as Re is increased to that of the full-scale prototype.

Predicted aerodynamic drag on the prototype:

$$F_{D,p} = \tfrac{1}{2}\rho_p V_p^2 A_p C_{D,p}$$

$$= \tfrac{1}{2}(1.184\ \mathrm{kg/m^3})(26.8\ \mathrm{m/s})^2[16^2(0.159\ \mathrm{m})(0.257\ \mathrm{m})](0.76)\left(\frac{1\ \mathrm{N}}{1\ \mathrm{kg{\cdot}m/s^2}}\right)$$

$$= \mathbf{3400\ N}$$

Flows with Free Surfaces

For the case of model testing of flows with free surfaces (boats and ships, floods, river flows, aqueducts, hydroelectric dam spillways, interaction of waves with piers, soil erosion, etc.), complications arise that preclude complete similarity between model and prototype.

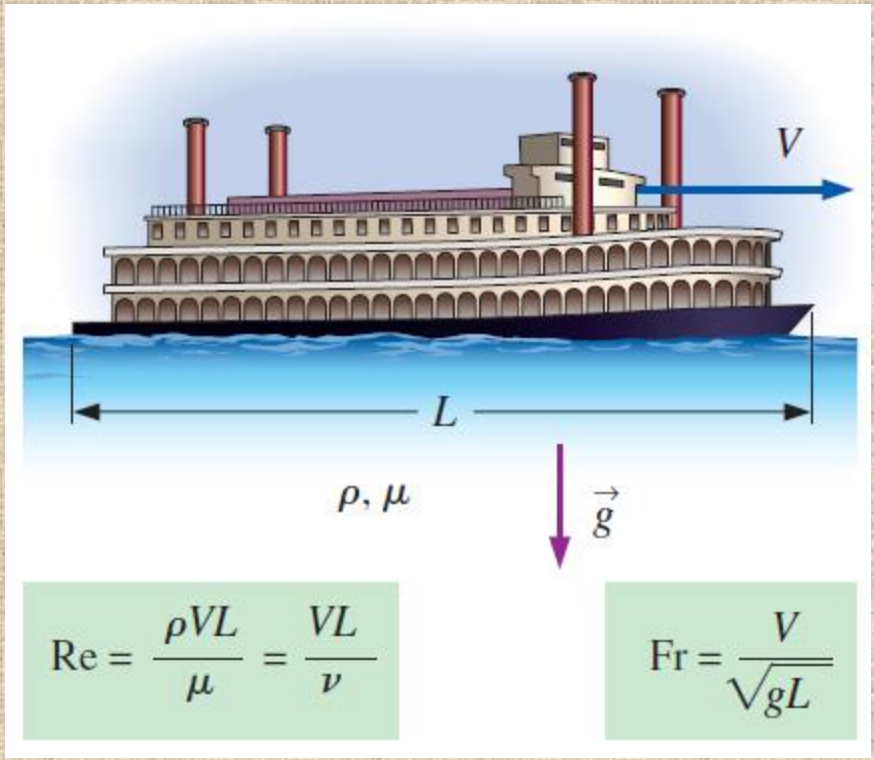
For example, if a model river is built to study flooding, the model is often several hundred times smaller than the prototype due to limited lab space.

Researchers often use a **distorted model** in which the vertical scale of the model (e.g., river depth) is exaggerated in comparison to the horizontal scale of the model (e.g., river width).

In addition, the model riverbed slope is often made proportionally steeper than that of the prototype.

These modifications result in incomplete similarity due to lack of geometric similarity.

Model tests are still useful under these circumstances, but other tricks (like deliberately roughening the model surfaces) and empirical corrections and correlations are required to properly scale up the model data.



$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}}$$

In many flows involving a liquid with a free surface, both the Reynolds number and Froude number are relevant nondimensional parameters. Since it is not always possible to match both Re and Fr between model and prototype, we are sometimes forced to settle for incomplete similarity.

$$\mathrm{Re}_p = \frac{V_p L_p}{\nu_p} = \mathrm{Re}_m = \frac{V_m L_m}{\nu_m}$$

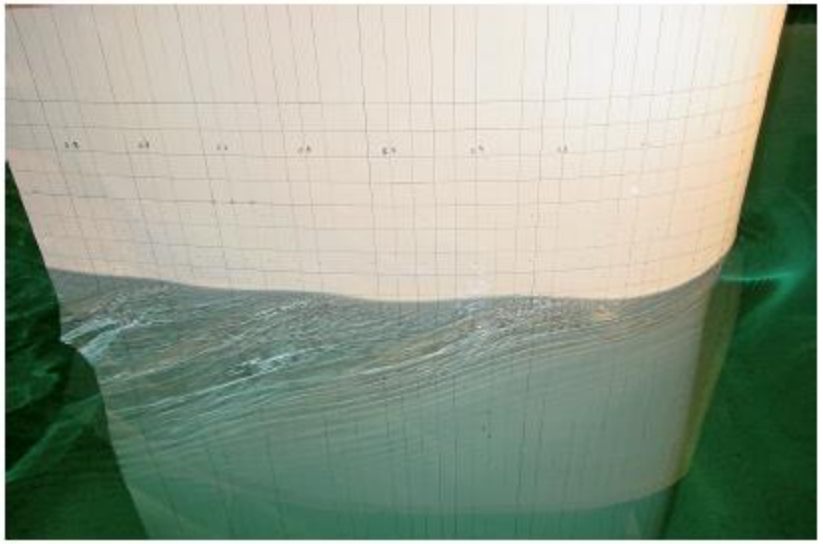
$$\mathrm{Fr}_p = \frac{V_p}{\sqrt{gL_p}} = \mathrm{Fr}_m = \frac{V_m}{\sqrt{gL_m}}$$

$$\frac{L_m}{L_p} = \frac{\nu_m}{\nu_p}\frac{V_p}{V_m} = \left(\frac{V_m}{V_p}\right)^2$$

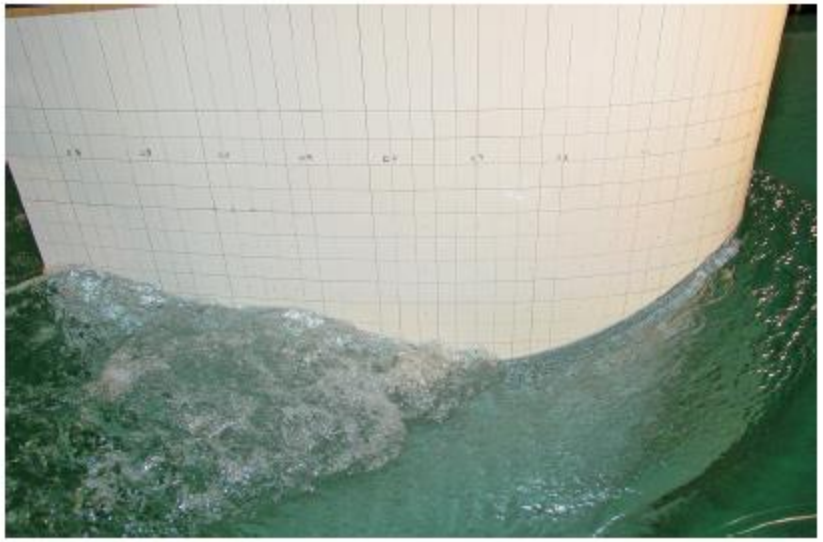
Required ratio of kinematic viscosities to match both Re and Fr:

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$

To ensure complete similarity we would need to use a liquid whose kinematic viscosity satisfies this equation.



(*a*)



(*b*)

(*c*)

A NACA 0024 airfoil being tested in a towing tank at Fr (*a*) 0.19, (*b*) 0.37, and (*c*) 0.55. In tests like this, the Froude number is the most important nondimensional parameter.

EXAMPLE 7–11 Model Lock and River

In the late 1990s the U.S. Army Corps of Engineers designed an experiment to model the flow of the Tennessee River downstream of the Kentucky Lock and Dam (Fig. 7–44). Because of laboratory space restrictions, they built a scale model with a length scale factor of $L_m/L_p = 1/100$. Suggest a liquid that would be appropriate for the experiment.

A 1:100 scale model constructed to investigate navigation conditions in the lower lock approach for a distance of 2 mi downstream of the dam. The model includes a scaled version of the spillway, powerhouse, and existing lock. In addition to navigation, the model was used to evaluate environmental issues associated with the new lock and required railroad and highway bridge relocations. The view here is looking upstream toward the lock and dam. At this scale, 52.8 ft on the model represents 1 mi on the prototype. A pickup truck in the background gives you a feel for the model scale.

Properties For water at atmospheric pressure and at $T = 20°C$, the prototype kinematic viscosity is $\nu_p = 1.002 \times 10^{-6}$ m²/s.

Analysis From Eq. 7–24,

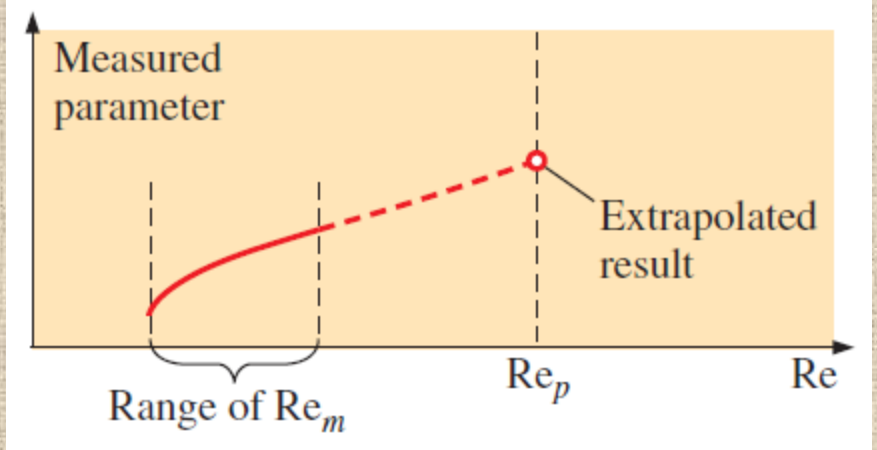
Required kinematic viscosity of model liquid:

$$\nu_m = \nu_p\left(\frac{L_m}{L_p}\right)^{3/2} = (1.002 \times 10^{-6}\,\mathrm{m^2/s})\left(\frac{1}{100}\right)^{3/2} = \mathbf{1.00 \times 10^{-9}\,m^2/s} \qquad (1)$$

Thus, we need to find a liquid that has a viscosity of 1.00×10^{-9} m²/s. A quick glance through the appendices yields no such liquid. Hot water has a lower kinematic viscosity than cold water, but only by a factor of about 3. Liquid mercury has a very small kinematic viscosity, but it is of order 10^{-7} m²/s—still two orders of magnitude too large to satisfy Eq. 1. Even if liquid mercury would work, it would be too expensive and too hazardous to use in such a test. What do we do? The bottom line is that *we cannot match both the Froude number and the Reynolds number in this model test.*

In other words, it is impossible to achieve complete similarity between model and prototype in this case. Instead, we do the best job we can under conditions of incomplete similarity. Water is typically used in such tests for convenience.

Discussion It turns out that for this kind of experiment, Froude number matching is more critical than Reynolds number matching. As discussed previously for wind tunnel testing, Reynolds number independence is achieved at high enough values of Re. Even if we are unable to achieve Reynolds number independence, we can often extrapolate our low Reynolds number model data to predict full-scale Reynolds number behavior (Fig. 7–45). A high level of confidence in using this kind of extrapolation comes only after much laboratory experience with similar problems.



In many experiments involving free surfaces, we cannot match both the Froude number and the Reynolds number. However, we can often *extrapolate* low Re model test data to predict high Re prototype behavior.

We mention the importance of similarity in the production of Hollywood movies in which model boats, trains, airplanes, buildings, monsters, etc., are blown up or burned.

Movie producers must pay attention to dynamic similarity in order to make the small-scale fires and explosions appear as realistic as possible.

You may recall some low-budget movies where the special effects are unconvincing.

In most cases this is due to lack of dynamic similarity between the small model and the full-scale prototype.

If the model's Froude number and/or Reynolds number differ too much from those of the prototype, the special effects don't look right, even to the untrained eye.

The next time you watch a movie, be on the alert for incomplete similarity!

Summary

- Dimensions and units
- Dimensional homegeneity
  - ✓ Nondimensionalization of Equations
  - ✓ Vapor Pressure and Cavitation
- Dimensional analysis and similarity
- The method of repeating variables and the Buckingham pi theorem
- Experimental testing, modeling and, incomplete similarity
  - ✓ Setup of an Experiment and Correlation of Experimental Data
  - ✓ Incomplete Similarity
  - ✓ Wind Tunnel Testing
  - ✓ Flows with Free Surfaces