Table 11.3. Output Data: Able's Observed Utilization by Replication and Run Length

Replication,	*Run Length, T_E*		
r	*2 Hours*	*4 Hours*	*8 Hours*
1	0.808	0.796	0.785
2	0.875	0.825	0.833
3	0.708	0.787	0.806
4	0.842	0.837	0.833
5	0.742	0.825	0.808
6	0.767	0.775	0.800
7	0.792	0.787	0.794
8	0.950	0.867	0.827
9	0.833	0.821	0.815
10	0.717	0.750	0.821
11	0.817	0.808	0.798
12	0.842	0.746	0.817
13	0.850	0.846	0.854
14	0.850	0.846	0.848
15	0.767	0.783	0.796
16	0.817	0.804	0.813

being simulated. Nevertheless, the data are presented for illustration purposes. Notice that for equal total run length over all replications ($R \times T_E$), such as 8 replications of 4 hours each (= 32 hours) or 4 replications of 8 hours each (= 32 hours), the point estimators have approximately equal precision. ◀

Table 11.4. Able's Estimated Utilization with Standard Error $\widehat{\rho} \pm \text{s.e.}(\widehat{\rho})$ for Various Run Lengths and Number of Replications

Number of Replications R	*Run Length, T_E*		
	2 Hours	*4 Hours*	*8 Hours*
4	0.808 ± 0.036	0.811 ± 0.011	0.814 ± 0.011
8	0.811 ± 0.027	0.812 ± 0.011	0.810 ± 0.006
16	0.811 ± 0.015	0.806 ± 0.009	0.816 ± 0.005

11.4.3 Confidence Intervals with Specified Precision

By expression (11.10), the half-length (h.l.) of a $100(1-\alpha)\%$ confidence interval for a mean θ, based on the t distribution, is given by

$$\text{h.l.} = t_{\alpha/2,R-1}\widehat{\sigma}(\widehat{\theta})$$

where $\widehat{\sigma}(\widehat{\theta}) = S/\sqrt{R}$, S is the sample standard deviation, and R is the number of replications. Suppose that an error criterion ϵ is specified; in other words, it is desired to estimate θ by $\widehat{\theta}$ to within $\pm\epsilon$ with high probability, say at least $1 - \alpha$. Thus it is desired that a sufficiently large sample size, R, be taken to satisfy

$$P(|\widehat{\theta} - \theta| < \epsilon) \geq 1 - \alpha$$

When the sample size, R, is fixed, as in Section 11.4.2, no guarantee can be given for the resulting error. But if the sample size can be increased, an error criterion can be specified.

Assume that an initial sample of size R_0 has been observed; that is, the simulation analyst initially makes R_0 independent replications. In practice, R_0 is 2 or larger, but at least 4 or 5 is recommended with 10 or more being desirable. The R_0 replications will be used to obtain an initial estimate S_0^2 of the population variance σ^2. To meet the half-length criterion, a sample size R must be chosen such that $R \geq R_0$ and

$$\text{h.l.} = \frac{t_{\alpha/2,R-1}S_0}{\sqrt{R}} \leq \epsilon \tag{11.20}$$

Solving for R in Inequality (11.20) shows that R is the smallest integer satisfying $R \geq R_0$ and

$$R \geq \left(\frac{t_{\alpha/2,R-1}S_0}{\epsilon}\right)^2 \tag{11.21}$$

Since $t_{\alpha/2,R-1} \geq z_{\alpha/2}$, an initial estimate for R is given by

$$R \geq \left(\frac{z_{\alpha/2}S_0}{\epsilon}\right)^2 \tag{11.22}$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentage point of the standard normal distribution. And since $t_{\alpha/2,R-1} \approx z_{\alpha/2}$ for large R (say $R \geq 50$), the second inequality for R is adequate when R is large. After determining the final sample size, R, collect $R - R_0$ additional observations (i.e., make $R - R_0$ additional replications, or start over and make R total replications) and form the $100(1 - \alpha)\%$ confidence interval for θ by

$$\widehat{\theta} - t_{\alpha/2,R-1}\frac{S}{\sqrt{R}} \leq \theta \leq \widehat{\theta} + t_{\alpha/2,R-1}\frac{S}{\sqrt{R}} \tag{11.23}$$

where $\widehat{\theta}$ and S are computed based on all R replications, $\widehat{\theta}$ by Equation (11.15) and $S^2/R = \widehat{\sigma}^2(\widehat{\theta})$ by Equation (11.16). The half-length of the confidence interval given by Inequality (11.23) should be approximately ϵ or smaller; however, with the additional $R - R_0$ observations, the variance estimator S^2 may differ somewhat from the initial estimate S_0^2, possibly causing the half-length to be greater than desired. If the confidence interval (11.23) is too large, the procedure may be repeated, using Inequality (11.21) to determine an even larger sample size.

Example 11.12

Suppose that it is desired to estimate Able's utilization in Example 11.7 to within ±0.04 with probability 0.95. An initial sample of size $R_0 = 4$ is taken, with the results given in Table 11.1 (also, Table 11.3 with $T_E = 2$ hours). An initial estimate of the population variance is $S_0^2 = R_0\widehat{\sigma}^2(\widehat{\rho}) = 4(0.036)^2 = 0.00518$. (See Example 11.10 or Table 11.4 for the relevant data. Notice how we recover S_0^2 from the standard error by squaring it and multiplying by the number of replications.) The error criterion is $\epsilon = 0.04$ and the confidence coefficient is $1 - \alpha = 0.95$. From Inequality (11.22), the final sample size must be at least as large as

$$\frac{z_{0.025}^2 S_0^2}{\epsilon^2} = \frac{(1.96)^2(0.00518)}{(0.04)^2} = 12.44$$

Next, Inequality (11.21) can be used to test possible candidates ($R = 13, 14, \ldots$) for final sample size, as follows:

R	13	14	15
$t_{0.025,R-1}$	2.18	2.16	2.14
$\frac{t_{0.025,R-1}^2 S_0^2}{\epsilon^2}$	15.39	15.10	14.83

Thus, $R = 15$ is the smallest integer satisfying Inequality (11.21), so $R - R_0 = 15 - 4 = 11$ additional replications are needed. Assuming that 12 additional replications (instead of 11) were made, with the output data $\widehat{\rho}_r$ as given in Table 11.3, the half-length for a 95% confidence interval would be given by $t_{0.025,15}$ s.e.$(\widehat{\rho}) = 2.13(0.015) = 0.032$, which is well within the precision criterion $\epsilon = 0.04$. The resulting 95% confidence interval for true utilization ρ is $0.811 - 0.032 = 0.779 \leq \rho \leq 0.843 = 0.811 + 0.032$. Notice the considerable improvement in precision (as measured by confidence interval half-length) compared to Example 11.10, which used a fixed sample size ($R = 4$). ◀

11.4.4 Confidence Intervals for Quantiles

To present the interval estimator for quantiles, it is helpful to review the interval estimator for a mean in the special case when the mean represents a proportion or probability, p. In this book we have chosen to treat a proportion or probability as just a special case of a mean. However, in many statistics texts probabilities are treated separately.

When the number of independent replications $Y_1, \ldots, Y_R$ is large enough so that $t_{\alpha/2,R-1} \doteq z_{\alpha/2}$, the confidence interval for a probability p is often