Another example of Seasonal Process

A producer of ice-cream thinks that with changing season, demand of ice-cream will change. That company wants to forecast demand for the next year (Season by season). Historical data of company, which reflect seasonality and trend, is given in the table.

Here, a year can be divided into four seasons, each three months or a quarter of a year. Many processes naturally have some number of seasons in a year. The methods presented here can be used for any season length.

To highlight the seasonality and trend, we plot by season in next figure $d_t = (a + bt)c_t + \varepsilon_t$

Where a= the constant portion, b= the slope of the trend component, c_t= the seasonal factor for period t, ε_t= the uncontrollable randomness.

Table: Quarterly demand of company

	Year			
Season	1	2	3	4
Fall	10	16	15	28
Winter	14	22	27	40
Spring	8	14	18	25
Summer	25	35	40	65

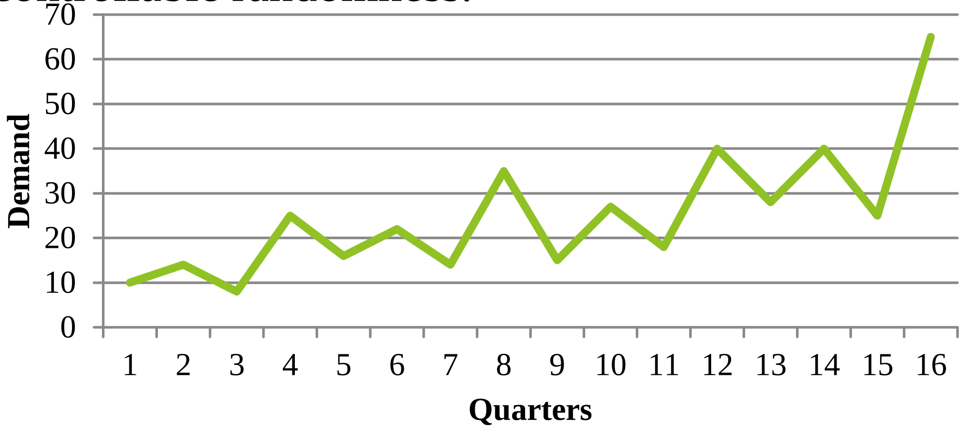


Figure: Seasonal data with seasonal trend

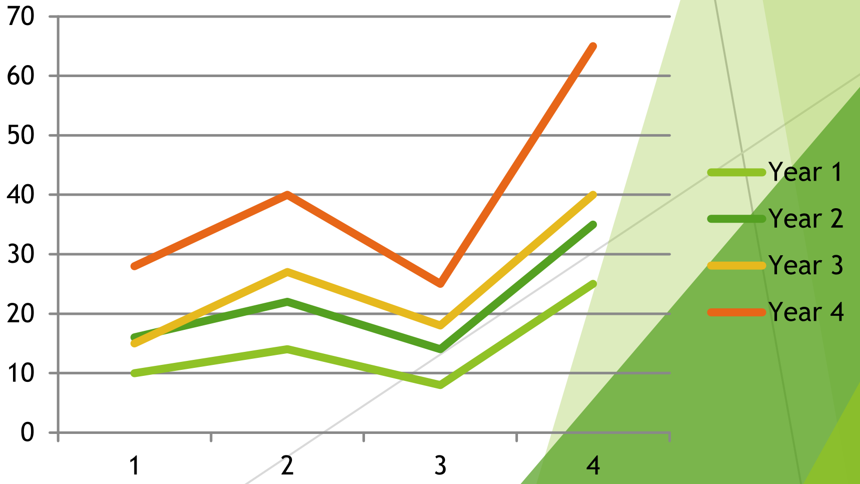


Figure: Seasonal data with trend plotted by season

The forecasting method is to estimate the parameters of the model and use them to generate the forecast. To forecast, we get initial estimates of the components of the model and update them using exponential smoothing.

Let: d_t = the demand in period t

L = length of season in a year (or other time frame)

T = the number of periods of data available

$T = mL$ where

m = the number of full seasons of data available

S_t = the estimate for the constant term a calculated at period t

b_t = the estimate for the trend term b calculated at time t

C_t = the estimate of the seasonal component for period t

Winter's Method:

Step 1: Calculate the average of each of the last two seasons of data, $\bar{d}_1$ and $\bar{d}_2$, then $b_t = \frac{\bar{d}_2 - \bar{d}_1}{L}$

Step 2: calculate the overall average $\bar{d} = \frac{1}{t}\sum_{t=1}^{n} d_t$

And use it and b_t to calculate the estimate constant S_t $\quad S_t = \bar{d} + (\frac{n-1}{2})b_t$

Step 3: Calculate the seasonal factors using the equation $C_t = \frac{1}{m}\sum_{i=0}^{m-1}\frac{d_{t-iL}}{S_t - (T - [t-iL])b_t}$

$$t = T - L + 1, T - L + 2, \dots, T$$

Step 4: To forecast k periods ahead, use $F_{t,t+k} = (S_t + kb_t)C_{t+k-gl}$,

g = the smallest integer greater than or equal to k/L

Apply winter method and forecast demands for periods of 9, 10, 11, and 12

Period	1	2	3	4	5	6	7	8	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
d_t	10	14	8	25	16	22	14	35	**15**	**27**	**18**	**40**	**28**	**40**	**25**	**65**

Period	d_t
(1 W) 1	15
(1 Sp) 2	27
(1 Su) 3	18
(1 F) 4	40
(2 W) 5	28
(2 Sp) 6	40
(2 Su) 7	25
(2 F) 8	65

Step 1:

$$\bar{d}_1 = \frac{15 + 27 + 18 + 40}{4} = 25$$

$$\bar{d}_2 = \frac{28 + 40 + 25 + 65}{4} = 39.5$$

$$b_t = \frac{\bar{d}_2 - \bar{d}_1}{L} \rightarrow b_8 = \frac{39.5 - 25}{4} = 3.625$$

Step 2:

$$\bar{d} = \frac{1}{t}\sum_{t=1}^{n} d_t = \frac{15 + 27 + 18 + 40 + 28 + 40 + 25 + 65}{8} = 32.25$$

$$S_t = \bar{d} + \left(\frac{n-1}{2}\right) b_t \rightarrow S_8 = 32.25 + \left(\frac{8-1}{2}\right) 3.625 = 44.9375$$

Step 3:

$$C_t = \frac{1}{m}\sum_{i=0}^{m-1}\frac{d_{t-iL}}{S_t-(T-[t-iL])b_t} \qquad t = T-L+1, T-L+2, \dots, T$$

$$C_5 = \frac{1}{2}\left(\frac{28}{44.9375-3(3.625)}+\frac{15}{44.9375-7(3.625)}\right) = 0.794396$$

$$C_6 = \frac{1}{2}\left(\frac{40}{44.9375-2(3.625)}+\frac{27}{44.9375-6(3.625)}\right) = 1.11289$$

$$C_7 = \frac{1}{2}\left(\frac{25}{44.9375-1(3.625)}+\frac{18}{44.9375-5(3.625)}\right) = 0.638236$$

$$C_8 = \frac{1}{2}\left(\frac{65}{44.9375-0(3.625)}+\frac{40}{44.9375-4(3.625)}\right) = 1.380311$$

Step 4:

$$F_{t,t+k} = (S_t + kb_t)C_{t+k-gL}$$

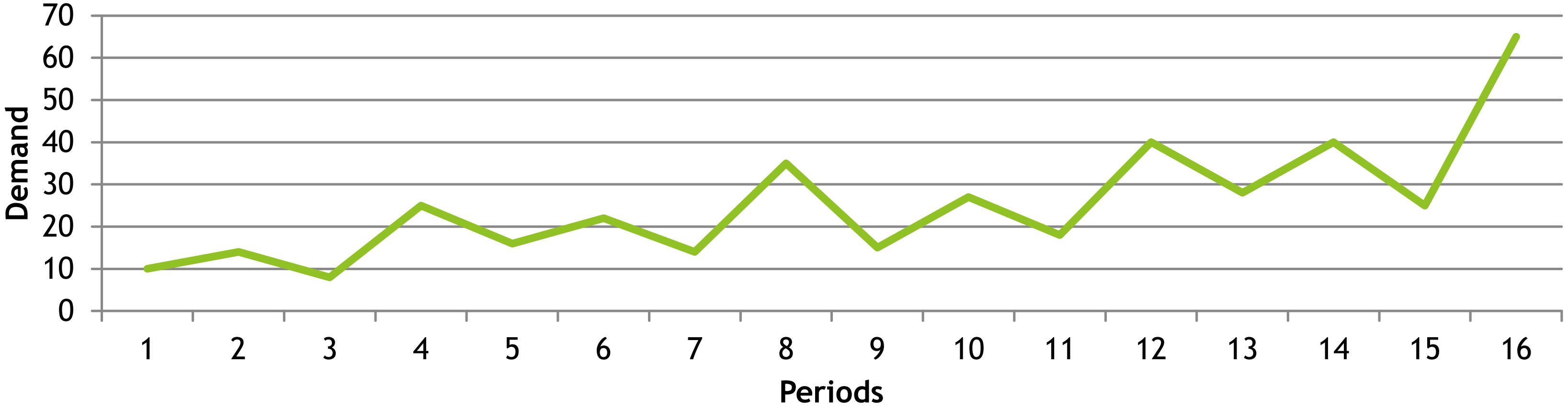
$$F_{16,17} = F_{8,9} = (S_8 + 1b_8)C_5 = (44.9375 + 1 * 3.625)0.794396 = 38.57785575$$

$$F_{16,18} = F_{8,10} = (S_8 + 2b_8)C_6 = (44.9375 + 2 * 3.625)1.11289 = 58.07894688$$

$$F_{16,19} = F_{8,11} = (S_8 + 3b_8)C_7 = (44.9375 + 3 * 3.625)0.638236 = 35.62154675$$

$$F_{16,20} = F_{8,12} = (S_8 + 4b_8)C_8 = (44.9375 + 4 * 3.625)1.380311 = 82.04223506$$

Example: What will happen if we ignore seasonality behavior of historical data? In that case we should focus on increasing trend and apply double exponential smoothing method for forecasting demand of periods 17, 18, 19, and 20.



Double exponential smoothing method is used when there is a trend in data. This method uses two smoothing parameters to update the level and trend components. In this forecasting process three equations are used: the first equation is for smoothing time series, the second equation is for smoothing trend, and the third equation is for the combination of above two equations. So we have:

$$S_t = \alpha d_t + (1-\alpha)(S_{t-1} + b_{t-1}) \qquad 0 < \alpha < 1$$
$$b_t = \beta(S_t - S_{t-1}) + (1-\beta)b_{t-1} \qquad 0 < \beta < 1$$
$$F_{t,t+k} = S_t + kb_t$$

Period	d_t	S_t	b_t	F_t
1	10			
2	14	10	4	
3	8	12.2	3.1	14
4	25	18.21	4.555	15.3
5	16	20.7355	3.54025	22.765
6	22	23.593	3.19889	24.2758
7	14	22.9543	1.2801	26.7919
8	35	27.4641	2.89493	24.2344
9	15	25.7513	0.59108	30.359
10	27	26.5397	0.68972	26.3424
11	18	24.4606	-0.6947	27.2294
12	40	28.6361	1.74042	23.7659
13	28	29.6636	1.38394	30.3765
14	40	33.7333	2.72681	31.0475
15	25	33.0221	1.0078	36.4601
16	65	43.3209	5.65332	34.0299

$$F_{t,t+k} = S_t + kb_t$$

$$F_{16,17} = 43.3209 + 1 * 5.65332 = 48.97422$$
$$F_{16,18} = 43.3209 + 2 * 5.65332 = 54.62755$$
$$F_{16,19} = 43.3209 + 3 * 5.65332 = 60.28087$$
$$F_{16,20} = 43.3209 + 4 * 5.65332 = 65.93419$$

S_1 and b_1 are not defined, and one way to identify these values is assuming that the initial value is equal to its expectations. Using this as starting point, set $S_2 = d_1 = 10$. Then subtract d_1 from d_2 to get b_2:$b_2 = d_2 - d_1 = 14 - 10 = 4$. Hence, at the end of period 2, forecast for the period 3 is $14(F_3 = 10 + 4)$. We assume $\alpha = 0.3\ and\ \beta = 0.5$ to solve this example. The actual demand for the period 3 is 8.

$$S_3 = 0.3 * 8 + (1 - 0.3)(10 + 4) = 12.2$$
$$b_3 = 0.5(12.2 - 10) + (1 - 0.5)4 = 3.1$$
$$F_{2,3} = 12.2 + 3.1 = 15.3$$
$$S_4 = 0.3 * 25 + (1 - 0.3)(12.2 + 3.1) = 18.21$$
$$b_4 = 0.5(18.21 - 12.2) + (1 - 0.5)4 = 4.555$$
$$F_{3,4} = 18.21 + 4.555 = 22.765$$
$$S_5 = 0.3 * 16 + (1 - 0.3)(18.21 + 4.555) = 20.7355$$
$$b_5 = 0.5(20.7355 - 18.21) + (1 - 0.5)4 = 4.555$$
$$F_{3,4} = 18.21 + 4.555 = 22.765$$

Simple way:

To implement double exponential smoothing at time t, we need values of S_{t-1} and b_{t-1}.

$$S_t = \alpha d_t + (1-\alpha)(S_{t-1} + b_{t-1}) \qquad 0 < \alpha < 1$$
$$b_t = \beta(S_t - S_{t-1}) + (1-\beta)b_{t-1} \qquad 0 < \beta < 1$$
$$F_{t,t+k} = S_t + k b_t$$

A simple way to do this is first divide the data into two equal groups, and compute the average of each. To convert this difference to an estimate of the slope, divide it by the number of time periods separating the two averages. Then to get an estimate of the intercept, use the overall average plus the slope estimate per period times the number of periods from the midpoint to the current period.

It means that $b_t = \frac{\bar{d}_2 - \bar{d}_1}{n/2}$ and $S_t = \bar{d} + \frac{n-1}{2} b_t$

Solution:

First, compute the averages of the months 1 to 8, and 9 to 16.

$$\bar{d}_1 = \frac{10 + 14 + 8 + 25 + 16 + 22 + 14 + 35}{8} = 18$$

$$\bar{d}_2 = \frac{15 + 27 + 18 + 40 + 28 + 40 + 25 + 65}{8} = 32.25$$

$$\bar{d} = \frac{10 + 14 + 8 + 25 + \cdots + 25 + 65}{16} = 25.125$$

$$b_t = \frac{\bar{d}_2 - \bar{d}_1}{n/2} \rightarrow b_{16} = \frac{32.25 - 18}{8} = 1.78125$$

$$S_t = \bar{d} + \frac{n-1}{2} b_t \rightarrow S_{16} = 25.125 + \frac{16-1}{2}(1.78125) = 38.484375$$

Once we have our initial values, we can forecast for future periods. The forecast for period 17 is

$$F_{16,17} = S_{16} + (1 * b_{16}) = 38.48375 + (1 * 1.78125) = 40.265625$$

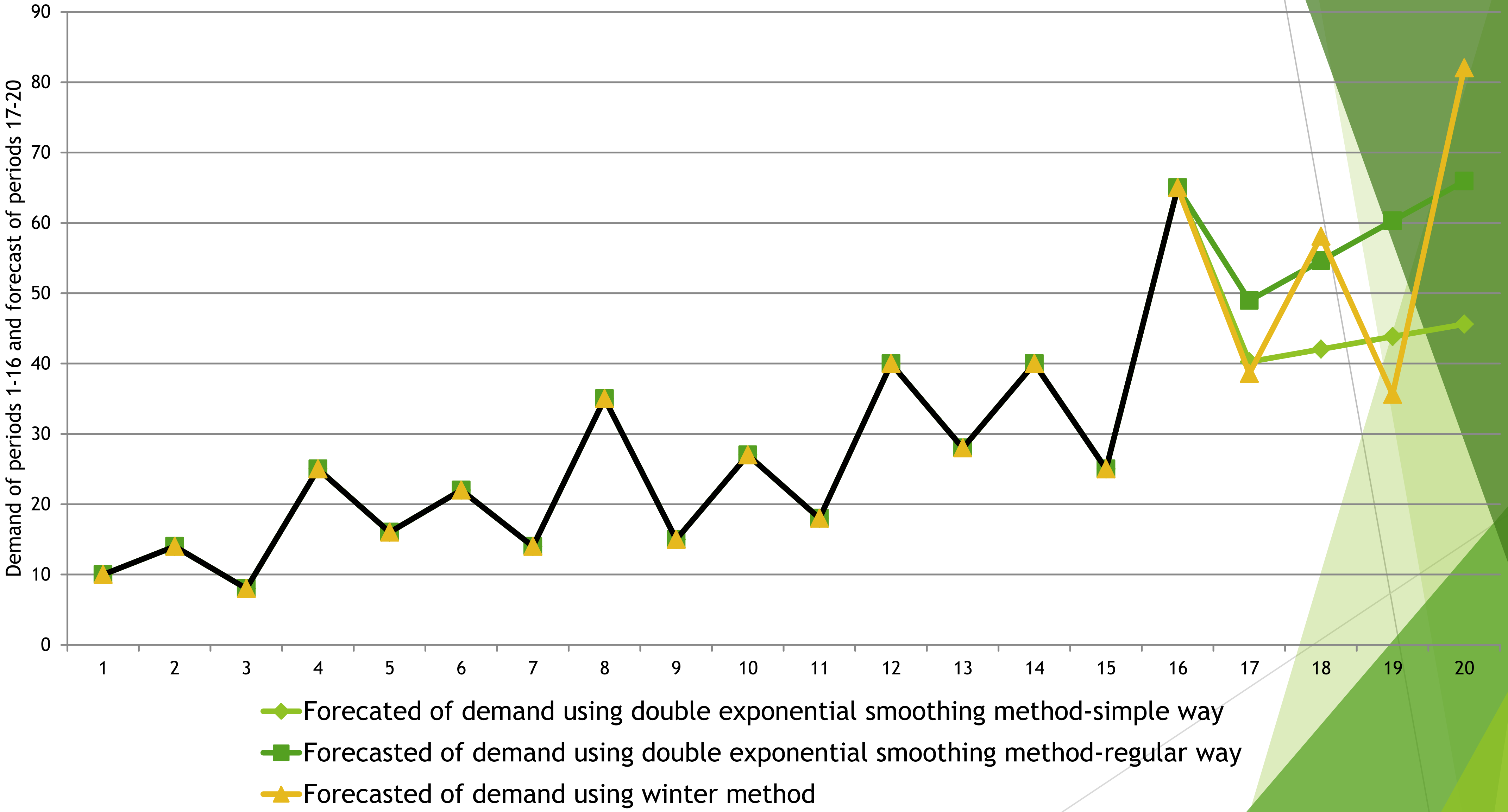
Similarly, forecasting for periods 18, 19, and 20

$$F_{16,18} = S_{16} + (2 * b_{16}) = 38.48375 + (2 * 1.78125) = 42.04625$$

$$F_{16,19} = S_{16} + (3 * b_{16}) = 38.48375 + (3 * 1.78125) = 43.8275$$

$$F_{16,20} = S_{16} + (4 * b_{16}) = 38.48375 + (4 * 1.78125) = 45.60875$$

Using the figure below, the forecasted values are compared using two methods (Double exponential smoothing (regular way and simple way) & winter)



To highlight the seasonality and trend, we plot by season in figure below. Demand for each quarter of the first year is below the demand of the same quarter for the second year, and the second year value is below the third year value, so demand appears to be growing. A good model must consider the constant portion of demand, the trend, and seasonality that shows with light green line. So in case of seasonality behavior in our historical data, we cannot ignore it.