Variance of a discrete random variable

If X is discrete random variable having a probability mass function $P(X = x_i)$, then $var(X) = E[(X - \mu)^2] = \sum_{i=1}^{n}(x_i - \mu)^2 P(X = x_i) = E(X^2) - (E(X))^2$

Example: Find $var(X)$, when the random variable X is such that

$$X = \begin{cases} 1 & with\ probability\ p \\ 0 & with\ probability\ 1 - p \end{cases}$$

x_i	1	0
$P(X = x_i)$	P	$1 - P$
$x_i P(X = x_i)$	P	0
$x_i^2 P(X = x_i)$	P	0

$$var(X) = E(X^2) - (E(X))^2 = \sum_{i=0}^{1} x_i^2 P(X = x_i) - \left(\sum_{i=0}^{1} x_i P(X = x_i)\right)^2 = P - P^2 = P(1 - P)$$

Properties of variance:

- $var(cX) = c^2.var(X)$ x is random variable and c is constant value
- $var(X + c) = var(X)$
- $var(X \pm Y) = var(X) + var(Y)$ X & Y are independent random variables
- $var[\sum_{i=1}^{n} x_i] = \sum_{i=1}^{n} var(x_i)$
- $Standard\ Deviation\ (SD) of\ X = \sqrt{var(X)}$

Example: The annual gross earnings of a soccer player are a random variable (Y) with expected value of 400,000 $ and SD of 80,000 $. The manager of soccer player receives 15% of this amount. Determine the E(X) & SD(X) of the amount received by the manager?

$E(Y) = 400,000$ $SD(Y) = 80,000$ $X = 0.15\ Y$

$E(X) = E(0.15\ Y) = 0.15\ E(Y) = 0.15 \times 400,000 = 60,000$

$Var(X) = Var(0.15\ Y) = 0.15^2\ Var(Y) = 0.15^2 \times 80,000^2 = 144,000,000$

$SD(X) = \sqrt{144,000,000} = 12000$

Example: in rolling a fair dice, where X is the side facing up, find $var(X)$?

x_i	1	2	3	4	5	6
$P(X = x_i)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$
$x_i P(X = x_i)$	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$
$x_i^2 P(X = x_i)$	$1/6$	$4/6$	$9/6$	$16/6$	$25/6$	$36/6$

$$E(X) = \sum_{i=1}^{n} x_i p(X = x_i) = 1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6 = 21/6$$

$$E(X^2) = \sum_{i=1}^{n} x_i^2 P(X = x_i) = 1/6 + 4/6 + 9/6 + 16/6 + 25/6 + 36/6 = 91/6$$

$$var(X) = E(X^2) - (E(X))^2 = \sum_{i=0}^{1} x_i^2 P(X = x_i) - \left(\sum_{i=0}^{1} x_i P(X = x_i)\right)^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{35}{12}$$

Example: $Y = 0.25X$, (X & Y are independent random variables). We know that
$\mu_X = 18$ & $\sigma_X^2 = 0{,}01$. Find mean and variance of Y?
$Y = 0.25X$
$\mu_Y = \mu_{0.25X} = 0.25\,\mu_Y = 0.25 \times 18 = 4.5$
$\sigma_Y^2 = \sigma_{0.25X}^2 = 0.25^2\sigma_X^2 = 0.25^2 \times 0.01 = 0.000625$
Example: for a constant C, P(X=C) =1. Find variance(X)

x_i	C
$P(X = x_i)$	1
$x_i P(X = x_i)$	C
${x_i}^2 P(X = x_i)$	C^2

$$var(X) = E(X^2) - \left(E(X)\right)^2 = \sum_{i=0}^{1} {x_i}^2 P(X = x_i) - \left(\sum_{i=0}^{1} x_i P(X = x_i)\right)^2 = C^2 - C^2 = 0$$

Continuous Random Variable:

A random variable is continuous if its probability is given by area under a curve. The curve is called a Probability Density Function (PDF) for the random variable.
Let X is a Continuous Random Variable with Probability Density Function f(X). Let a & b be any two numbers when a<b. then

- $p(a \le x \le b) = p(a < x < b) = p(a \le x < b) = p(a < x \le b) = \int_a^b f(x)dx$
- $p(a \le x) = p(a < x) = \int_a^{+\infty} f(x)dx$
- $p(x \le b) = p(x < b) = \int_{-\infty}^{b} f(x)dx$
- $\int_{-\infty}^{+\infty} f(X) = 1$

Expectation value and variance of a discrete random variable

$$E(X) = \int_{-\infty}^{+\infty} Xf(x)dx$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \int_{-\infty}^{+\infty} X^2 f(x)dx - \left[\int_{-\infty}^{+\infty} Xf(x)dx\right]^2$$

Example: A hole is drilled in a sheet metal and then a shaft is inserted through the hole. The shaft clearance is equal to difference between the radius of the hole and radius of the shaft. Let the random variable X denotes the clearance in millimeters. The probability density function of X is as below.

$$f(X) = \begin{cases} 1.25(1 - x^4) & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

What is the probability that the shaft clearance is larger than 0.8mm?

$$P(0.8 < X < 1) = \int_{0.8}^{1} 1.25(1 - x^4)dx = 1.25 \int_{0.8}^{1} (1 - x^4)dx = 1.25\left[x - \frac{x^5}{5}\right]\Big|_{0.8}^{1}$$
$$= 0.0819$$

What is the probability that the shaft clearance is larger than 0.8mm and smaller than 5?

$$P(0.8 < X < 5) = \int_{0.8}^{5} 1.25(1 - x^4)dx = 1.25 \int_{0.8}^{5} (1 - x^4)dx = 1.25\left[x - \frac{x^5}{5}\right]\Big|_{0.8}^{5} > 1$$

$$P(0.8 < X < 5) = \int_{0.8}^{1} 1.25(1 - x^4)dx + \int_{1}^{5} 1.25(1 - x^4)dx = 1.25 \int_{0.8}^{1} (1 - x^4)dx$$

$$= 1.25\left[x - \frac{x^5}{5}\right]\Big|_{0.8}^{1} = 0.0819$$

Example: The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & elsewhere \end{cases}$

Find the probability that over a period of one year, a family runs their vacuum cleaner

- Less than 120 hours
- Between 50 and 120 hours
- Expectation value of X
- Standard deviation of X

$$P(X < 1.2) = P(0 < X < 1) + P(1 < X < 1.2) = \int_{0}^{1} xdx + \int_{1}^{1.2} (2 - x)dx$$

$$= \left[\frac{x^2}{2}\right]\Big|_{0}^{1} + \left[2x - \frac{x^2}{2}\right]\Big|_{0}^{1.2} = 0.5 + 0.18 = 0.68$$

$$P(0.5 < X < 1.2) = \int_{0.5}^{1} xdx + \int_{1}^{1.2} (2 - x)dx = \left[\frac{x^2}{2}\right]\Big|_{0.5}^{1} = 0.5 - 0.125 = 0.375$$

$$E(X) = \int_{0}^{2} xf(x)dx = \int_{0}^{1} xxdx + \int_{1}^{2} x(2 - x)dx = \int_{0}^{1} x^2dx + \int_{1}^{2} (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]\Big|_{0}^{1} + \left[x^2 - \frac{x^3}{3}\right]\Big|_{1}^{2} = \frac{1}{3} + \left(4 - \frac{8}{3} - 1 + \frac{1}{2}\right) = 1$$

$$E(X^2) = \int_{0}^{2} x^2f(x)dx = \int_{0}^{1} x^2xdx + \int_{1}^{2} x^2(2 - x)dx = \int_{0}^{1} x^3dx + \int_{1}^{2} (2x^2 - x^3)dx$$

$$= \left[\frac{x^4}{4}\right]\Big|_{0}^{1} + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Big|_{1}^{2} = \frac{1}{4} + \left(\frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}\right) = \frac{7}{6}$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$$

$$SDx = \sqrt{Var(X)} = \sqrt{\frac{1}{6}} = 0.41$$

Example: consider the density function $f(x) = \begin{cases} k\sqrt{x} & 0 < x < 1 \\ 0 & elsewhere \end{cases}$

- Evaluate k
- Find the probability that X is between 0.3 and 1.6

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \rightarrow \int_{0}^{1} k\sqrt{x}\,dx = 1 \rightarrow k\int_{0}^{1} x^{1/2}\,dx = 1 \rightarrow = k\left[\frac{2x^{3/2}}{3}\right]\Big|_0^1 = 1 \rightarrow \frac{2k}{3} = 1 \rightarrow$$

$$k = \frac{3}{2}$$

$$P(0.3 < X < 1.6) = \int_{0.3}^{1} \frac{3}{2}x^{1/2}\,dx + \int_{1}^{1.6} \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}\left[\frac{2x^{3/2}}{3}\right]\Big|_{0.3}^{1} + 0 = 0.83$$