3-4 Confidence interval of $\boldsymbol{\mu_D = \mu_1 - \mu_2}$ for paired observations

İf $\bar{d}$ and s_d are the mean and standard deviation, respectively, of the normally distributed differences of n random pairs of measurements, a $100(1-\alpha)\%$ confidence interval for $\mu_D = \mu_1 - \mu_2$ Is given by

$$\bar{d} - t_{\alpha/2,n-1}\frac{s_d}{\sqrt{n}} < \mu_D < \bar{d} + t_{\alpha/2,n-1}\frac{s_d}{\sqrt{n}}$$

Where $t_{\alpha/2,n-1}$ is the t-value with n-1 (Degree of Freedom), leaving an area of $\alpha/2$ to the right.

For example Suppose a sample of n students were given a diagnostic test before studying a particular module and then again after completing the module. We want to find out if, in general, our teaching leads to improvements in students' knowledge/skills (i.e. test scores). We can use the results from our sample of students to draw conclusions about the impact of this module in general.

Example: using the above example with n=20 students, the following results were obtained

student	Pre-module score	Post-module score
1	18	22
2	21	25
3	16	17
4	22	24
5	19	16
6	24	29
7	17	20
8	21	23
9	23	19
10	18	20
11	14	15
12	16	15
13	16	18
14	19	26
15	18	18
16	20	24
17	12	18
18	22	25
19	15	19
20	17	16

- Find a 95% confidence interval for mean value of difference between post and pre-module score of these sample
- Can we claim that studying that module improved grades of students?

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{4+4+1+2-3+5+3+2-4+2+1-1+2+7+0+4+6+3+4-1}{20}$$

$$= \frac{41}{20} = 2.05$$

$$s_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}}$$

$$= \sqrt{\frac{(4-2.05)^2 + (4-2.05)^2 + (1-2.05)^2 + (2-1.05)^2 + (-3-2.05)^2 + (5-2.05)^2 + \cdots + (-1-2.05)^2}{20-1}}$$

$$= 2.837$$

$$p\left(\bar{d} - t_{\alpha/2, n-1}\frac{s_d}{\sqrt{n}} < \mu_D < \bar{d} + t_{\alpha/2, n-1}\frac{s_d}{\sqrt{n}}\right) = 1 - \alpha$$

$$p\left(2.05 - 2.093\frac{2.837}{\sqrt{20}} < \mu_D < 2.05 + 2.093\frac{2.837}{\sqrt{20}}\right) = 0.95$$

$$p(0.73 < \mu_D < 3.37) = 0.95$$

4. Confidence interval on proportion of one variable

4-1 Confidence interval on proportion of one variable

A point estımator of the proportion $P = X/_N$ in a binomial experiment is given by the statistic $\hat{p} = x/_n$, where x represents the number of successes in a trial. Therefore, the sample proportion $\hat{p} = x/_n$ will be used as the point estimate of the parameter P.

If $\hat{p}$ is the proportion of success in a random sample of size n and $\hat{p} = 1 - \hat{q}$, an approximate $100(1-\alpha)\%$ confidence interval, for the binomial parameter P is given by

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} < P < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$$

Where $z_{\alpha/2}$ is the z value leaving an area of $\alpha/2$ to the right .

Example: in a random sample of n=500 families owning television sets in the city of Hamilton, Canada, it is found that x=340 subscribe to HBO, find a 95% confidence interval for the actual proportion of families with television sets in this city that subscribe to HBO.

$$n = 500, x = 340, \hat{p} = \frac{x}{n} = \frac{340}{500} = 0.68$$

$$p\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} < P < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}\right) = 1 - \alpha$$

$$p\left(0.68 - 1.96\sqrt{\frac{0.68 \times 0.32}{500}} < P < 0.68 + 1.96\sqrt{\frac{0.68 \times 0.32}{500}}\right) = 0.95$$

$$p(0.68 - 0.0409 < P < 0.68 + 0.0409) = 0.95$$

$$p(0.6391 < P < 0.7209) = 0.95$$

Theorem: if $\bar{p}$ is used as an estimate of P, we can be $100(1-\alpha)\%$ confident that the error will not be exceed a specified amount $e$when the sample size is $n = (\frac{z_{\alpha/2}}{e})^2 \hat{p}\hat{q}$

Example: How large a sample is required if we want to be 95% confident that our estimate of P in the previous example is off by less than 0.02?

$$n = (\frac{z_{\alpha/2}}{e})^2 \hat{p}\hat{q} = (\frac{1.96}{0.02})^2 0.68 \times 0.32 = 2089.8 \cong 2090$$

4-2 Confidence interval on difference between two proportions

If $\hat{p}_1$ and $\hat{p}_2$are the proportions of successes in random samples of size n_1 and n_2, respectively, $\hat{q}_1 = 1 - \hat{p}_1$, and $\hat{q}_2 = 1 - \hat{p}_2$, an approximate $100(1-\alpha)\%$ confidence interval for the difference of two binomial parameters, $P_1 - P_2$, is given by

$$(\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} < P_1 - P_2 < (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

Where $z_{\alpha/2}$ is the z value leaving an area of $\alpha/2$ to the right .

Example: A certain change in a process for manufacturing component parts is being considered. Samples are taken under both the existing and the new process so as to determine if the new process results in an improvement. If 75 of 1500 items from the existing process are found to be defective and 80 of 2000 items from the new process are found to be defective.

- Find a 90% confidence interval for the true difference in the proportion of defectives between the existing and the new process.
- Find a 90% uper and lower bounds on the difference between proportion of defectives between the existing and the new process.

$$p\left((\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} < P_1 - P_2 < (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}\right) = 1 - \alpha$$

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{75}{1500} = 0.05, \qquad \hat{p}_2 = \frac{x_2}{n_2} = \frac{80}{2000} = 0.04, \quad \frac{\alpha}{2} = 0.05 \rightarrow z\alpha/_2 = z_{0.05} = 1.645$$

$$p\left((0.05 - 0.04) - 1.645\sqrt{\frac{0.05 \times 0.95}{1500} + \frac{0.04 \times 0.96}{2000}} < P_1 - P_2 \right.$$

$$\left. < (0.05 - 0.04) + 1.645\sqrt{\frac{0.05 \times 0.95}{1500} + \frac{0.04 \times 0.96}{2000}} \right) = 0.90$$

$$p((0.01 - 0.0117) < P_1 - P_2 < (0.01 + 0.0117)) = 0.90$$

$$p(-0.0017 < P_1 - P_2 < 0.0217) = 0.90$$

$$upper: p\left(P_1 - P_2 < (\hat{p}_1 - \hat{p}_2) + z_\alpha\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} \right) = 1 - \alpha$$

$$p\left(P_1 - P_2 < (0.05 - 0.04) + 1.28\sqrt{\frac{0.05 \times 0.95}{1500} + \frac{0.04 \times 0.96}{2000}} \right) = 0.9$$

$$p(P_1 - P_2 < 0.033458) = 0.9$$

$$lower: p\left(P_1 - P_2 > (\hat{p}_1 - \hat{p}_2) - z_\alpha\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} \right) = 1 - \alpha$$

$$p\left(P_1 - P_2 > (0.05 - 0.04) - 1.28\sqrt{\frac{0.05 \times 0.95}{1500} + \frac{0.04 \times 0.96}{2000}} \right) = 0.9$$

$$p(P_1 - P_2 > -0.013458) = 0.9$$

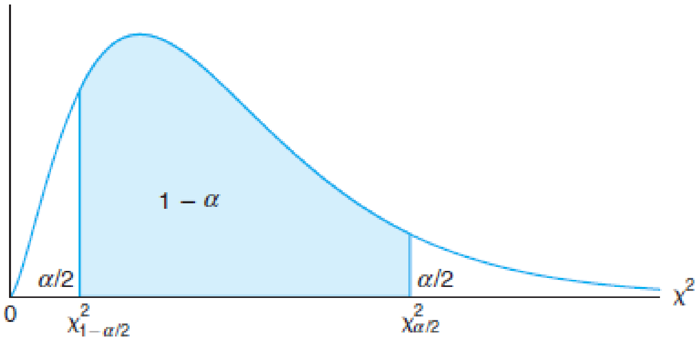
5. Single sample – estimating variance

If a sample of size n is drawn from a normal population with variance σ^2 and the sample variance s^2 is computed, we obtain value of statisticss^2. This computed sample variance is used as a point estimate of σ^2. Hence, the statistics s^2 is called an estimator of σ^2.

An interval estimate of σ^2can be established by using the statistics

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

The statistics χ^2 has a chi-square distribution with n-1 degrees of freedom when samples are chosen from a normal population. We may write (see figure)



$$P\left[\chi^2_{1-\alpha/2} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2}\right] = 1-\alpha$$

If s^2 is the variance of a random sample of size n from a normal distribution, a $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

Where $\chi^2_{1-\alpha/2}$ and $\chi^2_{\alpha/2}$ are values of the chi-squared distribution with n-1 degrees of freedom, leaving areas of $1-\alpha/2$ and $\alpha/2$, respectively, to the right.

Example: the following are the weights, in diagrams, of 10 packages of grass seed distributed by a certain company: 46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, and 46.0.

- Find a 95% confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal distribution.
- Find upper and lower side of confidence interval for that variance.

$$p\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right) = 1-\alpha$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{46.4+46.1+45.8+47+46.1+45.9+45.8+46.9+45.2+46}{10} = \frac{461.2}{10}$$

$$= 46.12$$

$$s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

$$= \frac{(46.4-46.12)^2+(46.1-46.12)^2+(45.8-46.12)^2+(47-46.12)^2+(46.1-46.12)^2+\cdots+(46-46.12)^2}{20-1}$$

$$= 0.286$$

$$p\left(\frac{(10-1)0.286}{19.023} < \sigma^2 < \frac{(10-1)0.286}{2.7}\right) = 0.95 \rightarrow p(0.135 < \sigma^2 < 0.953) = 0.95$$

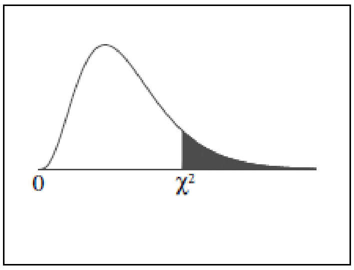
$$upper: p\left(\sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha}}\right) = 1-\alpha$$

$$p\left(\sigma^2 < \frac{(10-1)0.286}{3.235}\right) = 0.95 \rightarrow p(\sigma^2 < 0.789571) = 0.95$$

$$lower: p\left(\sigma^2 > \frac{(n-1)s^2}{\chi^2_{\alpha}}\right) = 1-\alpha$$

$$p\left(\sigma^2 > \frac{(10-1)0.286}{16.919}\right) = 0.95 \rightarrow p(\sigma^2 > 0.152137) = 0.95$$

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

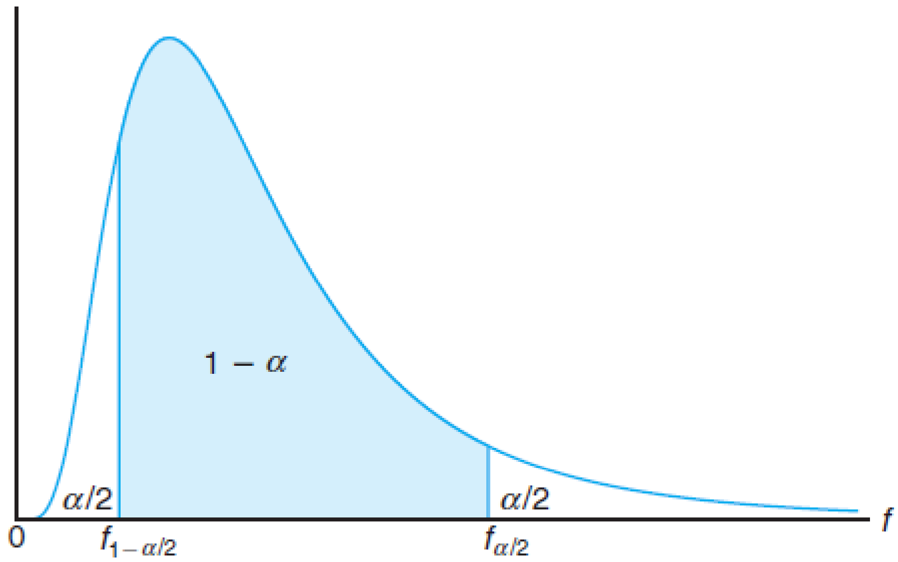
6. Two Samples – Estimating the Ratio of Two Variance

A point estimator of the ratio of two population variances ${\sigma_1^2}/{\sigma_2^2}$ is given by the ratio ${s_1^2}/{s_2^2}$ of the sample variances. Hence, the statistics ${s_1^2}/{s_2^2}$ is called an estimator of ${\sigma_1^2}/{\sigma_2^2}$. If σ_1^2 and σ_2^2 are the variances of normal populations, we can establish an interval estimate of ${\sigma_1^2}/{\sigma_2^2}$ by using the statistics $F = \frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2}$

The random variable F has an F-distribution with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees if freedom. Therefore, we may write (see figure)

$$P\left(f_{(1-{\alpha}/{2}),(v_1,v_2)} < F < f_{({\alpha}/{2}),(v_1,v_2)}\right) = 1 - \alpha$$

Where $f_{({\alpha}/{2}),(v_1,v_2)}$ and $f_{({\alpha}/{2}),(v_1,v_2)}$ is an F-value with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom, leaving an area of ${\alpha}/{2}$ to the right.



Substituting for F, we write

$$P\left(f_{(1-{\alpha}/{2}),(v_1,v_2)} < \frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} < f_{({\alpha}/{2}),(v_1,v_2)}\right) = 1 - \alpha$$

Multiplying each term in the inequality by ${s_1^2}/{s_2^2}$ and then inverting each term, we obtain

$$P\left(\frac{s_1^2}{s_2^2}\frac{1}{f_{({\alpha}/{2}),(v_1,v_2)}} < {\sigma_1^2}/{\sigma_2^2} < \frac{s_1^2}{s_2^2}\frac{1}{f_{(1-{\alpha}/{2}),(v_1,v_2)}}\right) = 1 - \alpha$$

We can replace the quantity $f_{(1-\alpha/2),(v_1,v_2)}$ by $\frac{1}{f_{(\alpha/2),(v_2,v_1)}}$, therefore

$$P\left(\frac{s_1^2}{s_2^2}\frac{1}{f_{(\alpha/2),(v_1,v_2)}} < \sigma_1^2/\sigma_2^2 < \frac{s_1^2}{s_2^2} f_{(\alpha/2),(v_2,v_1)}\right) = 1-\alpha$$

Where $f_{(\alpha/2),(v_1,v_2)}$ is an F-value with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right, and $f_{(\alpha/2),(v_2,v_1)}$ is a similar F-value with $v_2 = n_2 - 1$ and $v_1 = n_1 - 1$ degrees of freedom.

Example: A machine is used to fill bottles with vegetable oil. Two random samples are selected from the filled bottles and the oil is weighted in each bottle.

First sample	15.66	15.66	15.70	15.70	15.68	15.70
Second sample	15.78	15.7	15.78	15.79		

- Construct a 90% confidence interval on σ_1^2/σ_2^2
- Find upper and lower bound for this ratio with 99% confidence coefficient

$$P\left(\frac{s_1^2}{s_2^2}\frac{1}{f_{(\alpha/2),(v_1,v_2)}} < \sigma_1^2/\sigma_2^2 < \frac{s_1^2}{s_2^2} f_{(\alpha/2),(v_2,v_1)}\right) = 1-\alpha$$

$$\bar{x}_1 = \frac{\sum_{i=1}^{n_1} x_i}{n_1} = \frac{15.66 + 15.66 + 15.70 + 15.70 + 15.68 + 15.70}{6} = \frac{94.1}{6} = 15.68$$

$$s_1^2 = \frac{\sum_{i=1}^{n_1}(x_i - \bar{x})^2}{n_1 - 1}$$
$$= \frac{(15.66-15.68)^2 + (15.66-15.68)^2 + (15.70-15.68)^2 + (15.70-15.68)^2 + (15.68-15.68)^2 + (15.70-15.68)^2}{6-1}$$
$$= 0.0003867$$

$$\bar{x}_2 = \frac{\sum_{i=1}^{n_2} x_i}{n_2} = \frac{15.78 + 15.7 + 15.78 + 15.79}{4} = \frac{63.05}{4} = 15.76$$

$$s_2^2 = \frac{\sum_{i=1}^{n_2}(x_i - \bar{x})^2}{n_2 - 1} = \frac{(15.78-15.76)^2 + (15.70-15.76)^2 + (15.78-15.76)^2 + (15.79-15.76)^2}{4-1}$$
$$= 0.00009167$$

$$P\left(\frac{0.0003867}{0.00009167}\times\frac{1}{9.01} < \sigma_1^2/\sigma_2^2 < \frac{0.0003867}{0.00009167}\times 5.41\right) = 0.95$$

$$\rightarrow P\left(0.46819 < \sigma_1^2/\sigma_2^2 < 22.8215\right) = 0.95$$

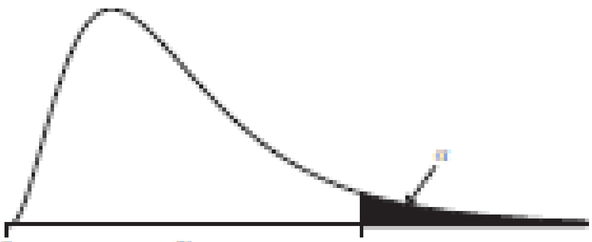
$$upper: P\left({\sigma_1^2}/{\sigma_2^2} < \frac{s_1^2}{s_2^2} f_{(\alpha),(v_2,v_1)} \right) = 1 - \alpha$$

$$upper: P\left({\sigma_1^2}/{\sigma_2^2} < \frac{0.0003867}{0.00009167} \times 12.06 \right) = 0.99 \rightarrow P\left({\sigma_1^2}/{\sigma_2^2} < 50.87381 \right) = 0.99$$

$$lower: P\left({\sigma_1^2}/{\sigma_2^2} > \frac{s_1^2}{s_2^2} \frac{1}{f_{(\alpha),(v_1,v_2)}} \right) = 1 - \alpha$$

$$lower: P\left({\sigma_1^2}/{\sigma_2^2} > \frac{0.0003867}{0.00009167} \times \frac{1}{28.24} \right) = 0.99 \rightarrow P\left({\sigma_1^2}/{\sigma_2^2} > 0.149376 \right) = 0.99$$

TABLE D: *F* Distribution



	$\alpha = .05$									
	df_1									
df_2	1	2	3	4	5	6	8	12	24	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	238.9	243.9	249.0	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.37	19.41	19.45	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	2.95	2.79	2.61	2.40
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
13	4.67	3.80	3.41	3.18	3.02	2.92	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.38	2.20	2.00	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.36	2.18	1.98	1.73
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
26	4.22	3.37	2.98	2.74	2.59	2.47	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.30	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.44	2.29	2.12	1.91	1.65
29	4.18	3.33	2.93	2.70	2.54	2.43	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.02	1.83	1.61	1.25
∞	3.84	2.99	2.60	2.37	2.21	2.09	1.94	1.75	1.52	1.00

Source: From Table V of R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*, published by Longman Group Ltd., London, 1974. (Previously published by Oliver & Boyd, Edinburgh.) Reprinted by permission of the authors and publishers.

TABLE D: (continued)

	$\alpha = .01$									
	df_1									
df_2	1	2	3	4	5	6	8	12	24	∞
1	4052	4999	5403	5625	5764	5859	5981	6106	6234	6366
2	98.49	99.01	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	34.12	30.81	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
11	9.65	7.20	6.22	5.67	5.32	5.07	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.20	4.86	4.62	4.30	3.96	3.59	3.16
14	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.79	3.45	3.08	2.65
18	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.51	3.17	2.80	2.36
22	7.94	5.72	4.82	4.31	3.99	3.76	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.23	2.90	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.20	2.87	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.66	2.34	1.95	1.38
∞	6.64	4.60	3.78	3.32	3.02	2.80	2.51	2.18	1.79	1.00