MATH103

Mathematics for Business and Economics – I

Linear Inequality, Absolute Value Inequality, Quadratic Inequality, Rational Inequality

LINEAR INEQUALITIES

An **inequality** is a statement that one algebraic expression is less than, or is less than or equal to, another algebraic expression.

If the equality symbol = in a linear equation is replaced by an inequality symbol (<, >, ≤, or ≥), the resulting expression is called a **first-degree, or linear, inequality**. For example

$$5 \le (1-3x)2 + \frac{x}{2}$$

is a linear inequality.

A **linear inequality in one variable** is an inequality that is equivalent to one of the forms

$$ax + b < 0 \quad \text{or} \quad ax + b \le 0,$$

where a and b represent real numbers and $a \neq 0$.

Rules for Inequalities

1. If the same number is added to or subtracted from both sides of an inequality, the resulting inequality has the same sense as the original inequality. Symbolically,

$$\text{if} \quad a < b, \quad \text{then} \quad a + c < b + c \quad \text{and} \quad a - c < b - c$$

For example, $7 < 10$, so $7 + 3 < 10 + 3$.

2. If both sides of an inequality are multiplied or divided by the same *positive* number, the resulting inequality has the same sense as the original inequality. Symbolically,

$$\text{if} \quad a < b \quad \text{and} \quad c > 0, \quad \text{then} \quad ac < bc \quad \text{and} \quad \frac{a}{c} < \frac{b}{c}$$

For example, $3 < 7$ and $2 > 0$, so $3(2) < 7(2)$ and $\frac{3}{2} < \frac{7}{2}$.

3. If both sides of an inequality are multiplied or divided by the same *negative* number, then the resulting inequality has the *reverse* sense of the original inequality. Symbolically,

$$\text{if} \quad a < b \quad \text{and} \quad c > 0, \quad \text{then} \quad a(-c) > b(-c) \quad \text{and} \quad \frac{a}{-c} > \frac{b}{-c}$$

For example, $4 < 7$ but $4(-2) > 7(-2)$ and $\frac{4}{-2} > \frac{7}{-2}$.

4. If both sides of an inequality are positive and we raise each side to the same positive power, then the resulting inequality has the same sense as the original inequality. Thus, if $0 < a < b$ and $n > 0$, then

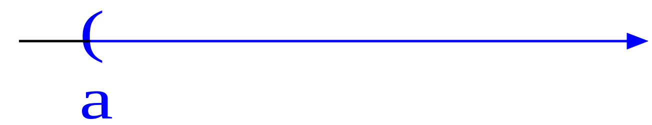
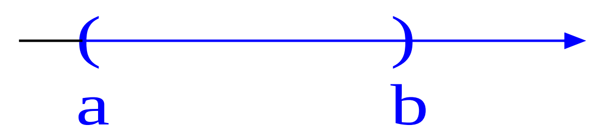
$$a^n < b^n \quad \text{and} \quad \sqrt[n]{a} < \sqrt[n]{b}$$

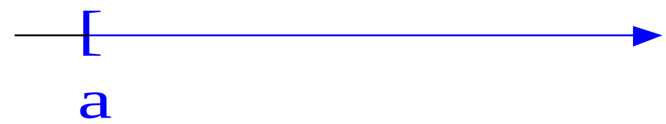
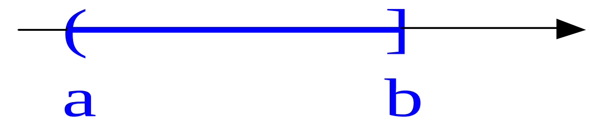
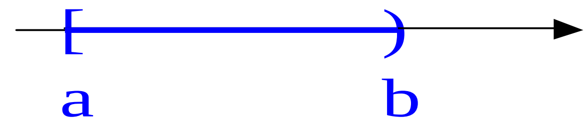
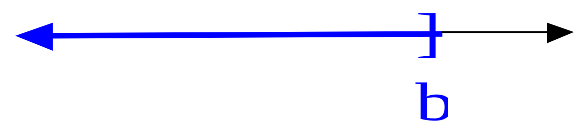
where we assume that n is a positive integer in the latter inequality. For example, $4 < 9$, so $4^2 < 9^2$ and $\sqrt{4} < \sqrt{9}$.

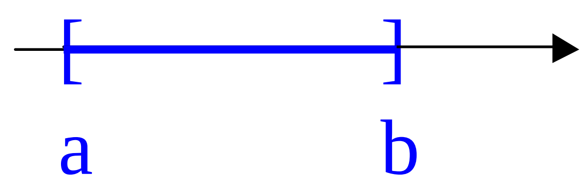
Linear Inequalities in One Variable

Interval Notation is used to write solution sets of inequalities.

Note: A parenthesis is used to indicate an endpoint in not included. A square bracket indicates the endpoint is included.

Interval Notation			
Type of Interval	**Set**	**Interval Notation**	**Graph**
Open Interval	{x| a < x }	(a, ∞)	
	{x | a < x < b}	(a, b)	
	{x | x < b}	(-∞, b)	
	{x | x is a real number}	(-∞, ∞)	

Interval Notation			
Type of Interval	**Set**	**Interval Notation**	**Graph**
Half-open Interval	$\{x \mid a \leq x\}$	$[a, \infty)$	
	$\{x \mid a < x \leq b\}$	$(a, b]$	
	$\{x \mid a \leq x \leq b\}$	$[a, b)$	
	$\{x \mid x \leq b\}$	$(-\infty, b]$	

Interval Notation			
Type of Interval	**Set**	**Interval Notation**	**Graph**
Closed Interval	$\{x \mid a \leq x \leq b\}$	$[a, b]$	

- Example 1: Solve $k - 5 > 1$

  $k - 5 + 5 > 1 + 5$

  $k > 6$

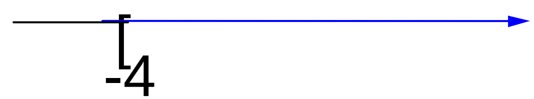
  Solution set: $(6, \infty)$

- Example 2: Solve $5x + 3 \geq 4x - 1$ and graph the solution set.

  $5x - 4x \geq -1 - 3$

  $x \geq -4$

  Solution set: $[-4, \infty)$



- Example 3: Solve $-2x < 10$

  $x > -5$

  Solution set: $(-5, \infty)$

- Example 4: Solve $2x < -10$

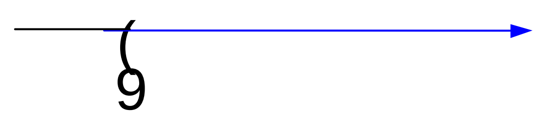
  $x < -5$

  Solution set: $(-\infty, -5)$

- Example 5: Solve $-9m < -81$ and graph the solution set

  $m > 9$

  Solution set: $(9, \infty)$



EXAMPLE 6 Solving and Graphing Linear Inequalities

Solve the inequality $3(x-1) < 5(x + 2) - 5$

Solution:

$3(x-1) < 5(x + 2) - 5$

$3x - 3 < 5x + 10 - 5$ Distribute the 3 and the 5

$3x - 3 < 5x + 5$ Combine like terms.

$-2x < 8$ Subtract 5x from both sides, and add 3 to both sides

$x > -4$ Notice that the sense of the inequality reverses when we divide both sides by -2.

$x > -4$ is equivalent to $(-4, \infty)$

EXAMPLE 7 Solving and Graphing Linear Inequalities

Solve the inequality $2(x\text{-}3) < 4$

Solution:

$$2(x-3) < 4$$

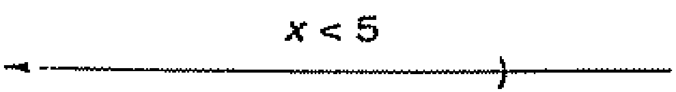
$$2x-6 < 4$$

$$2x-6+6 < 4+6$$

$$2x < 10$$

$$\frac{2x}{2} < \frac{10}{2}$$

$$x < 5$$



Line Notation

Solution Set = { x: x < 5 }

Set Notation

(-∞, 5)

Interval Notation

EXAMPLE 7 Solving and Graphing Linear Inequalities

Solve the inequality $3-2x \le 6$

Solution:

$$3 - 2x \le 6$$

$$-2x \le 3$$

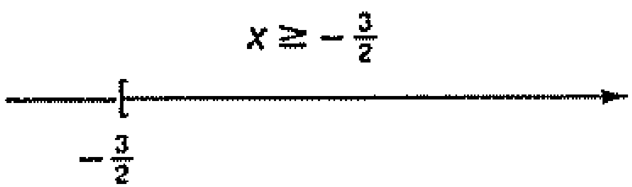
$$x \ge -\frac{3}{2}$$

The solution is $x \ge -\frac{3}{2}$

interval notation, $[-\frac{3}{2}, \infty)$.

Set Notation

Solution Set = { x: x >= 5 }



Line Notation

EXAMPLE 8 | Solving and Graphing Linear Inequalities

Solve the inequality $2(x-4)-3 > 2x-1$

Solution:

$$2(x-4)-3 > 2x-1$$

$$2x-8-3 > 2x-1$$

$$-11 > -1$$

Since it is never true that $-11 > -1$, there is no solution, and the solution set is $\emptyset$.

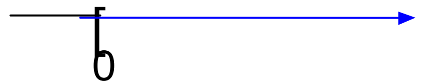
Example 9: Solve $6(x-1) + 3x \geq -x - 3(x + 2)$ and graph the solution set

Step 1: $6x-6 + 3x \geq -x - 3x - 6$

$9x - 6 \geq -4x - 6$

Step 2: $13x \geq 0$

Step 3: $x \geq 0$ Solution set: $[0, \infty)$



Example 10: Solve $\frac{1}{4}(m+3)+2 \leq \frac{3}{4}(m+8)$ and graph the solution set

$$\frac{1}{4}m+\frac{3}{4}+2 \leq \frac{3}{4}m+6$$

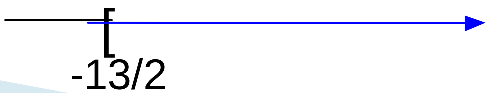
$$\frac{1}{4}m+\frac{11}{4} \leq \frac{3}{4}m+\frac{24}{4}$$

$$\frac{1}{4}m-\frac{3}{4}m \leq \frac{24}{4}-\frac{11}{4}$$

$$-\frac{2}{4}m \leq \frac{13}{4}$$

$$m \geq -\frac{13}{2}$$

Solution set: $[-\frac{13}{2}\ \infty)$



Solve the inequality $\frac{3}{2}(s-2)+1>-2(s-4)$

Solution:

$$\frac{3}{2}(s-2)+1 > -2(s-4)$$

$$2\left[\frac{3}{2}(s-2)+1\right] > 2[-2(s-4)]$$

$$3(s-2)+2 > -4(s-4)$$

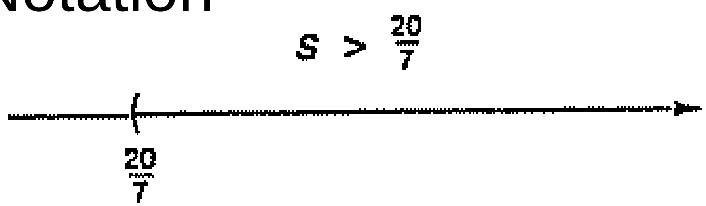
$$3s-4 > -4s+16$$

$$7s > 20$$

$$s > \frac{20}{7}$$

The solution is $(\frac{20}{7}, \infty)$;

Line Notation



The interval $(\frac{20}{7}, \infty)$.

ABSOLUTE VALUE INEQUALITY

Inequality	Solution
$\|x\| < d$	$-d < x < d$
$\|x\| \le d$	$-d \le x \le d$
$\|x\| > d$	$x < -d$ or $x > d$
$\|x\| \ge d$	$x \le -d$ or $x \ge d$

Ex: Solve the following inequalities.

a) $|x-2|<4$

$-4<x-2<4$

$-2<x<6$

$S=(-2,6)$

b) $|3-2x|\le 5$

$-5\le 3-2x\le 5$

$-8\le -2x\le 2$

$4\ge x\ge -1$

$S=[-1,4]$

c) $|x+5| \geq 7$

$x+5 \leq -7$ or $x+5 \geq 7$

$x \leq -12$ or $x \geq 2$

$S = (-\infty, -12] \cup [2, \infty)$

d) $|3x-1| < 5$

$-5 < 3x - 1 < 5$

$-4 < 3x < 6$

$-\frac{4}{3} < x < 2$

$S = \left(-\frac{4}{3}, 2\right)$

e) $|2x-5|\geq 3$

$2x-5\geq 3$ or $2x-5\leq -3$

$x\geq 4$ or $x\leq 1$

$S=(-\infty,1]\cup[4,\infty)$

f) $|4x-3|\leq -2$ $S=\varnothing$

Solving a Quadratic Inequality

$$ax^2 + bx + c \le 0, \qquad ax^2 + bx + c \ge 0$$

Given $\quad f(x) = ax^2 + bx + c$

Find the roots by equating to zero, $\quad ax^2 + bx + c = 0$

Solve by using either Factoring or Quadratic formula

Suppose x_1 and x_2 are the roots

x	$-\infty$	x_1		x_2	$+\infty$
$f(x)$	Same sign with a	0	opposite sign wth a	0	Same sign with a

Suppose $x_1 = x_2$ are the roots

x	$-\infty$	$x_1 = x_2$	$+\infty$
$f(x)$	Same sign with a	0	Same sign with a

Solving Quadratic Inequalities: 1st Method

Ex: Solve $x^2 + 3x - 10 < 0$

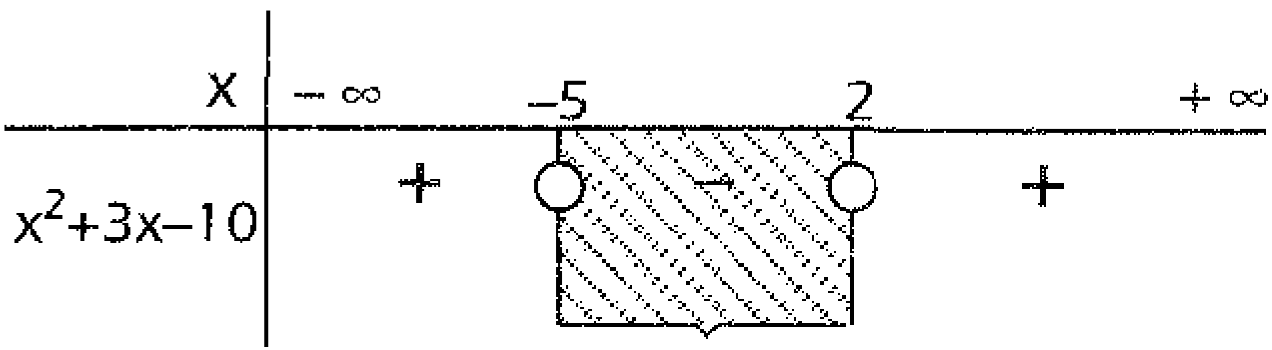
Solution :

$$x^2 + 3x - 10 = 0$$

$$\Rightarrow (x + 5)(x - 2) = 0$$

$$\Rightarrow x = -5, \quad x = 2$$

$$f(x) = x^2 + 3x - 10 \Rightarrow a = 1 > 0$$



Solution set $= (-5, 2)$

Solving Quadratic Inequalities: 2nd Method

$Solve\ x(x-1)(x+4) \leq 0$

Solution: If $f(x) = x(x-1)(x+4)$, then f is a polynomial function and is continuous everywhere. The zeros of f are 0, 1, and -4,

$$(-\infty, -4) \quad (-4, 0) \quad (0, 1) \quad (1, \infty)$$

Now, at a test point in each interval, we find the sign of $f(x)$:

$$f(-5) = (-)(-)(-) = - \quad \text{so } f(x) < 0 \text{ on } (-\infty, -4)$$

$$f(-2) = (-)(-)(+) = + \quad \text{so } f(x) > 0 \text{ on } (-4, 0)$$

$$f\left(\frac{1}{2}\right) = (+)(-)(+) = - \quad \text{so } f(x) < 0 \text{ on } (0, 1)$$

$$f(2) = (+)(+)(+) = + \quad \text{so } f(x) > 0 \text{ on } (1, \infty)$$

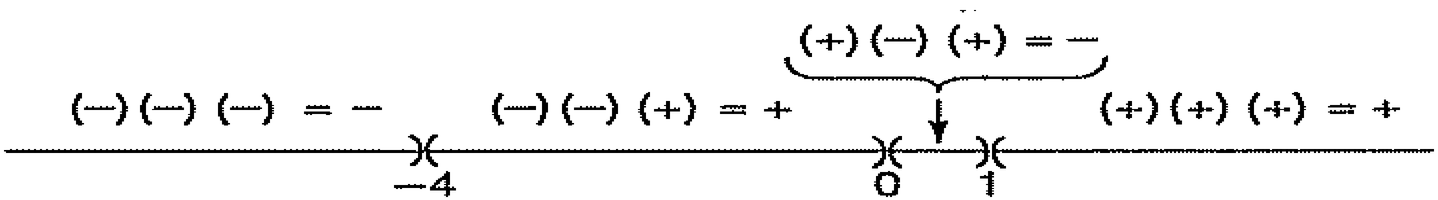


Figure shows the sign chart for $f(x)$. Thus, $x(x-1)(x+4) \leq 0$ on $(-\infty, -4]$ and $[0, 1]$.

Example : Consider the inequality $x^2 - x - 6 > 0$

Solution: We can find the values where the quadratic equals zero by solving the equation,

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

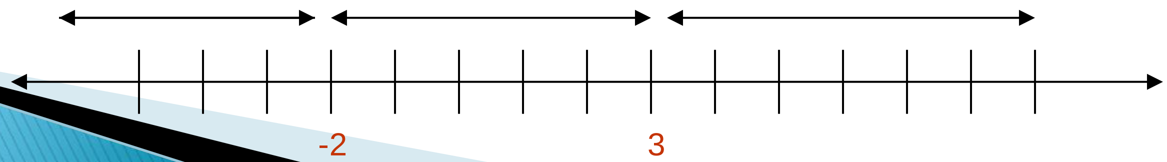
$$x - 3 = 0 \text{ or } x + 2 = 0$$

$$x = 3 \text{ or } x = -2$$

For the quadratic inequality, $x^2 - x - 6 > 0$

we found zeros 3 and –2 by solving the equation

$x^2 - x - 6 = 0$. Put these values on a number line and we can see three intervals that we will test in the inequality. We will test one value from each interval.



Interval	Test Point	Evaluate in the inequality	True/False
$(-\infty,-2)$	$x=-3$	$x^2-x-6>0$ $(-3)^2-(-3)-6=9+3-6=6>0$	True
$(-2,3)$	$x=0$	$x^2-x-6>0$ $(0)^2-(0)-6=0+0-6=-6<0$	False
$(3,\infty)$	$x=4$	$x^2-x-6>0$ $(4)^2-(4)-6=16-4-6=6>0$	True

Thus the intervals $(-\infty,-2)$ or $(3,\infty)$ make up the solution set for the quadratic inequality, $x^2-x-6>0$.

In summary, one way to solve quadratic inequalities is to find the zeros and test a value from each of the intervals surrounding the zeros to determine which intervals make the inequality true.

Example: Solve $2x^2 - 3x + 1 \le 0$

Solution: First find the zeros by solving the equation,

$$2x^2 - 3x + 1 = 0$$

$$2x^2 - 3x + 1 = 0$$

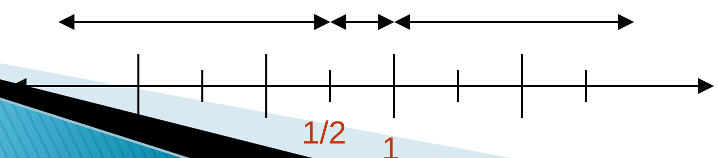
$$(2x-1)(x-1) = 0$$

$$2x - 1 = 0 \text{ or } x - 1 = 0$$

$$x = \frac{1}{2} \text{ or } x = 1$$

Now consider the intervals around the zeros and test a value from each interval in the inequality.

The intervals can be seen by putting the zeros on a number line.



Interval	Test Point	Evaluate in Inequality	True/False
$\left(-\infty, \frac{1}{2}\right)$	$x=0$	$2x^2-3x+1<0$ $2(0)^2-3(0)+1=0-0+1=1>0$	False
$\left(\frac{1}{2}, 1\right)$	$x=\frac{3}{4}$	$2x^2-3x+1<0$ $2\left(\frac{3}{4}\right)^2-3\left(\frac{3}{4}\right)+1=\frac{9}{8}-\frac{9}{4}+1=\frac{-1}{8}<0$	True
$(1, \infty)$	$x=2$	$2x^2-3x+1<0$ $2(2)^2-3(2)+1=8-6+1=3>0$	False

Thus the interval $\left(\frac{1}{2}, 1\right)$ makes up the solution set for

the inequality $2x^2-3x+1\leq 0$.

Ex: Solve $x^2 - 2x < -8$

Solution :

$$x^2 - 2x < -8 \Rightarrow x^2 - 2x + 8 < 0$$

$$x^2 - 2x + 8 = 0 \Rightarrow \Delta = 4 - 32 = -28$$

There are no real roots, thus

Solution set is empty set, {}

Ex: Solve $(2-x)(x^2+4)(x^2-3x) < 0$

Solution :

$$2 - x = 0 \Rightarrow x = 2$$

$x^2 + 4 = 0 \Longrightarrow$ **No real roots**

$$x^2 - 3x = 0 \Rightarrow x = 0,\ x = 3$$

x	$-\infty$	0		2		3	$+\infty$
$2-x$	+		+	○	−		−
x^2+4	+		+		+		+
x^2-3x	+	○	−		−	○	+
$f(x)$	+	○	−	○	+	○	−
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Solution set = $0 < x < 2$ veya $3 < x$

Practice Problems

$x^2 + 5x - 24 \leq 0$	$5x^2 - 13x + 6 < 0$
$12 - x - x^2 > 0$	$9 - x^2 \leq 0$
$3x^2 + 5x + 2 < 0$	$2x^2 - 5x + 1 < 0$
$16x^2 - 1 \geq 0$	$x^2 + 5x < -4$
$3x^2 + 2x + 1 > 0$	$x^2 \leq 2x - 4$

RATIONAL INEQUALITIES

Example: $$\frac{x^2 - x - 6}{x^2 + 4x - 5} \geq 0$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, x = -2$$

$$x^2 + 4x - 5 = 0$$

$$(x+5)(x-1) = 0$$

$$x = -5, x = 1$$

$$x = 3, x = -2, x = -5, x = 1$$

These are roots of the expressions.

Use Sign Table,

x	−5	−2	1	3	
x+5	−	+	+	+	+
x+2	−	−	+	+	+
x−1	−	−	−	+	+
x−3	−	−	−	−	+
f(x)	+	−	+	−	+

$$f(x)=\frac{x^2-x-6}{x^2+4x-5}\geq 0$$

$$S=(-\infty-5)\cup[-2,1)\cup[3,\infty)$$

Example: $\dfrac{x^2-4}{x}<0$

$$x^2-4=(x-2)(x+2)=0,\quad x=-2, x=2$$

$x=0, x=-2, x=2$ are the roots of these expressions.

Use Sign table,

x	−2		0	2
x+2	−	+	+	+
x	−	−	+	+
x−2	−	−	−	+
f(x)	−	+	−	+

$$S=(-\infty,-2)\cup(0,2)$$

Example: $\dfrac{x^2-1}{x^2+4}<0$

$$x^2-1=(x-1)(x+1)=0$$

$$x=-1, x=1$$

$$x^2+4=0, \text{ there is no real roots.}$$

X		−1	1
x+1	−	+	+
x−1	−	−	+
f(x)	+	−	+

$$S=(-1,1)$$

Example: $\dfrac{x^2+1}{x^2-4}<0$

$x^2+1=0$ there is no real roots.

$x^2-4=(x-2)(x+2)=0, \quad x=-2, x=2$

x	−2	2	
x+2	−	+	+
x−2	−	−	+
f(x)	+	−	+

$$S=(-2,2)$$

Example: $\dfrac{x^2+1}{x^2+4} > 0$ $\quad S = (-\infty, \infty)$

$x^2 + 1 = 0$ (always positive) there is no real roots.

$x^2 + 4 = 0$ (always positive) there is no real roots.

Example: $\dfrac{x^2+1}{x^2+4} < 0$ $\quad S = \varnothing$

Exercise: $\dfrac{x^2-1}{\left(4-x^2\right)\left(x^2-9\right)} \geq 0$