MATH103

Mathematics for Business and Economics – I

Linear Functions, Graphs in Rectangular Coordinates and Lines

Graphs in Rectangular Coordinates

An **ordered pair** of real numbers is a pair of real numbers in which the order is specified, and is written by enclosing a pair of numbers in parentheses and separating them with a comma.

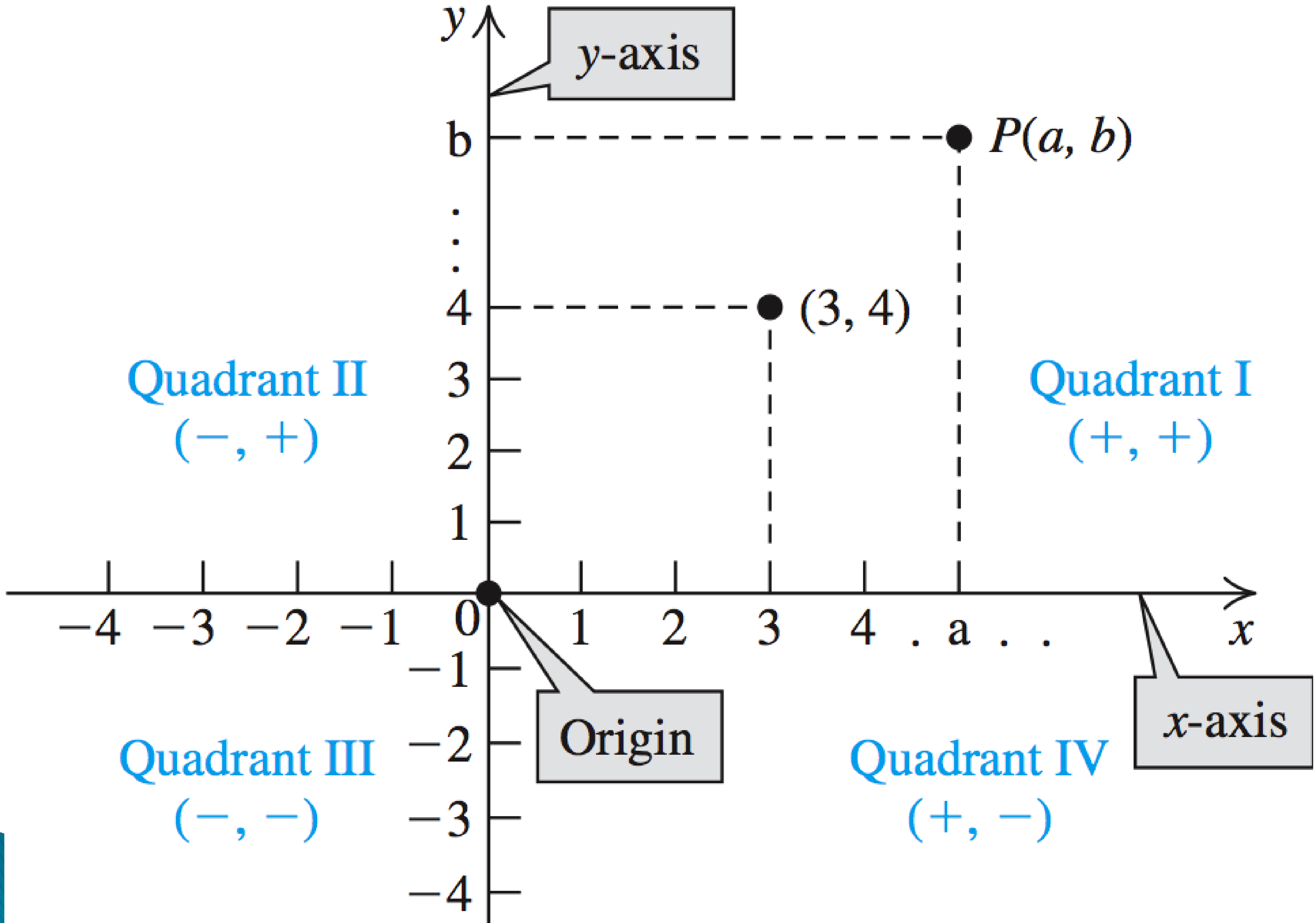
The ordered pair (a, b) has **first component** a and **second component** b. Two ordered pairs (x, y) and (a, b) are **equal** if and only if $x = a$ and $y = b$.

The sets of ordered pairs of real numbers are identified with points on a plane called the **coordinate plane** or the **Cartesian plane**.

Definitions

We begin with two coordinate lines, one horizontal (***x*-axis**) and one vertical (***y*-axis**), that intersect at their zero points. The point of intersection of the *x*-axis and *y*-axis is called the **origin**. The *x*-axis and *y*-axis are called coordinate axes, and the plane formed by them is sometimes called the ***xy*-plane**.

The axes divide the plane into four regions called **quadrants**, which are numbered as shown in the next slide. The points on the axes themselves do not belong to any of the quadrants.



y
y-axis
b
P(a, b)
4
(3, 4)
3
2
1
Quadrant II
(−, +)
Quadrant I
(+, +)
−4 −3 −2 −1 0 1 2 3 4 . a . .
x
−1
Origin
x-axis
−2
Quadrant III
(−, −)
Quadrant IV
(+, −)
−3
−4

Definitions

The figure shows how each ordered pair (a, b) of real numbers is associated with a unique point in the plane P, and each point in the plane is associated with a unique ordered pair of real numbers. The first component, a, is called the ***x*-coordinate** of P and the second component, b, is called the ***y*-coordinate** of P, since we have called our horizontal axis the x-axis and our vertical axis the y-axis.

Definitions

The *x*-coordinate indicates the point's distance to the right of, left of, or on the *y*-axis. Similarly, the *y*-coordinate of a point indicates its distance above, below, or on the *x*-axis. The signs of the *x*- and *y*-coordinates are shown in the figure for each quadrant. We refer to the point corresponding to the ordered pair (a, b) as the **graph of the ordered pair** (a, b) in the coordinate system. The notation $P(a, b)$ designates the point *P* in the coordinate plane whose *x*-coordinate is *a* and whose *y*-coordinate is *b*.

EXAMPLE 1 Graphing Points

Graph the following points in the *xy*-plane:

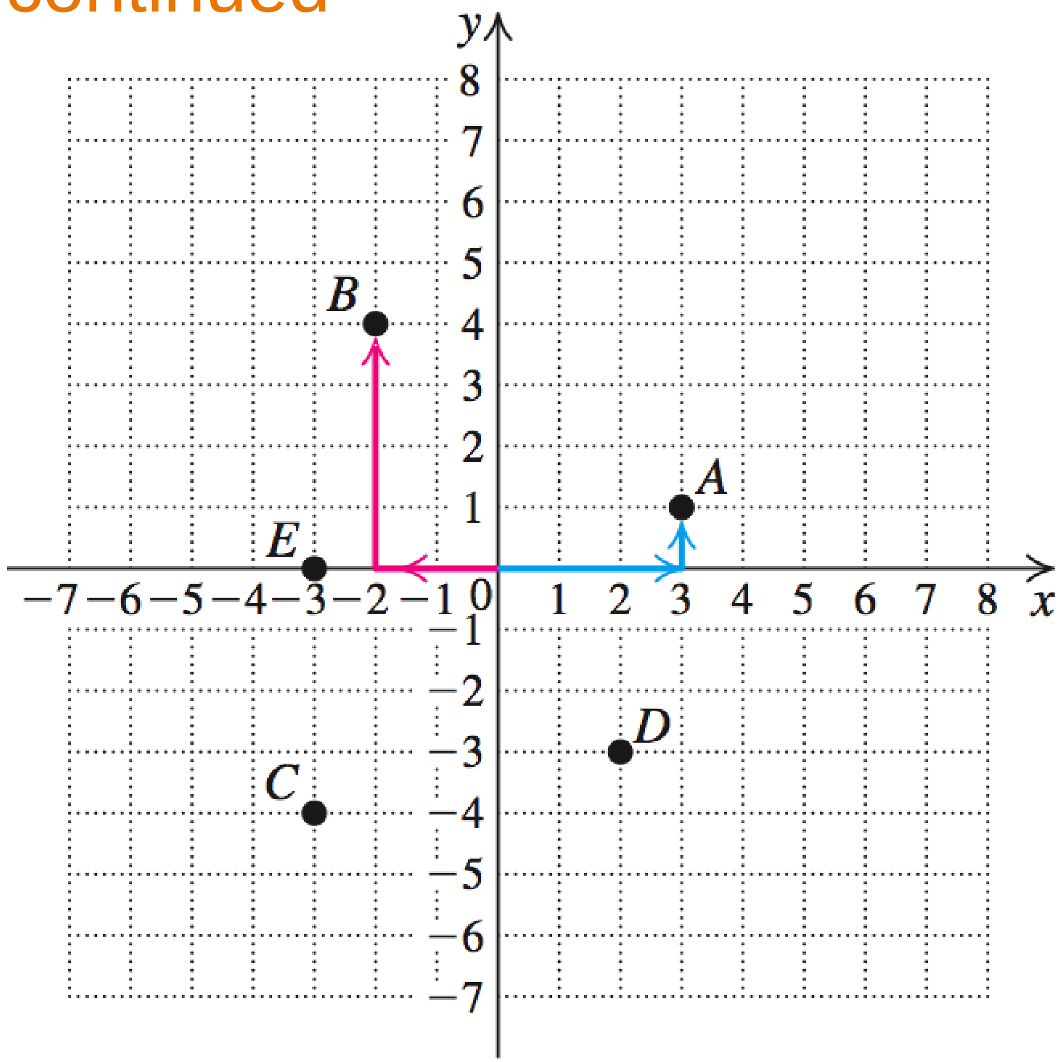
$$A(3,1),\ B(-2,4),\ C(-3,-4),\ D(2,-3),\ E(-3,0)$$

Solution

$A(3,1)$	3 units right, 1 unit up
$B(-2,4)$	2 units left, 4 units up
$C(-3,-4)$	3 units left, 4 units down
$D(2,-3)$	2 units right, 3 units down
$E(-3,0)$	3 units left, 0 units up or down

EXAMPLE 1 Graphing Points

Solution continued



Definitions

The points where a graph intersects (crosses or touches) the coordinate axes are of special interest in many problems. Since all points on the *x*-axis have a *y*-coordinate of 0, any point where a graph intersects the *x*-axis has the form (*a*, 0). The number *a* is called an ***x*-intercept** of the graph. Similarly, any point where a graph intersects the *y*-axis has the form (0, *b*), and the number *b* is called a ***y*-intercept** of the graph.

PROCEDURE FOR FINDING THE INTERCEPTS OF A GRAPH

Step1 To find the *x*-intercepts of an equation, set $y = 0$ in the equation and solve for *x*.

Step 2 To find the *y*-intercepts of an equation, set $x = 0$ in the equation and solve for *y*.

EXAMPLE 1 Finding Intercepts

Find the x- and y-intercepts of the graph of $y = 2x + 3$.

Solution: If $y = 0$, then

$$0 = 2x + 3 \quad \text{so that} \quad x = -\frac{3}{2}$$

Thus, the x-intercept is $(-\frac{3}{2}, 0)$. If $x = 0$, then

$$y = 2(0) + 3 = 3$$

EXAMPLE 2 Finding Intercepts

Find the x- and y-intercepts of the graph of the equation $y = x^2 - x - 2$.

Solution

Step 1 To find the x-intercepts, set $y = 0$, solve for x.

$$0 = x^2 - x - 2$$

$$0 = (x+1)(x-2)$$

$$x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -1 \quad \text{or} \quad x = 2$$

The x-intercepts are –1 and 2.

EXAMPLE 2 Finding Intercepts

Solution continued

Step 2 To find the *y*-intercepts, set $x = 0$, solve for *y*.

$$y = 0^2 - 0 - 2$$

$$y = -2$$

The *y*-intercept is –2.

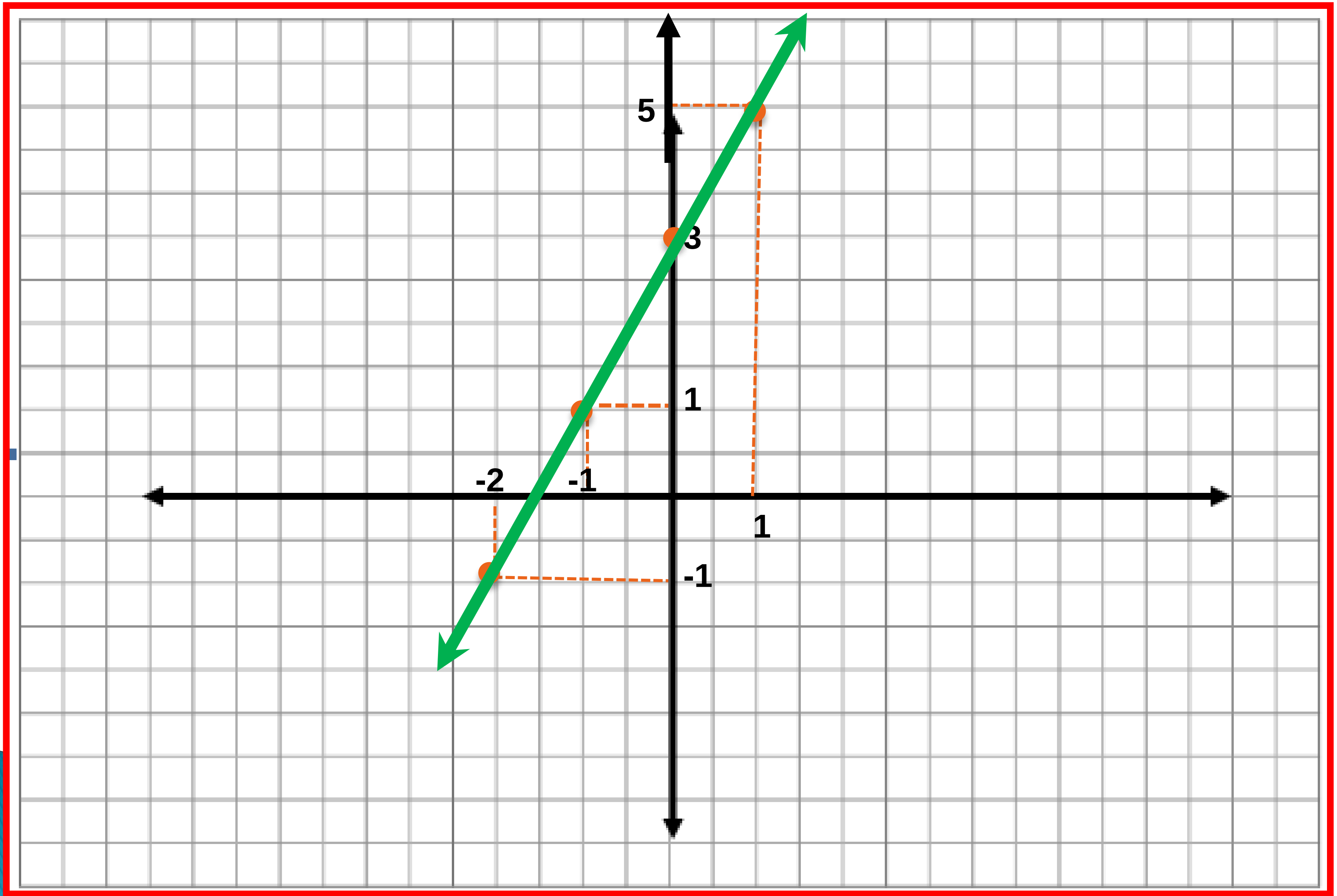
The graph of the linear equation(Line)

The following steps can be used to draw the graph of a linear equation.

Step1) Select at least 2 values for x
Step2) Substitute them in the equation and find the corresponding values for y
Step3) Plot the points on cartesian plane
Step4) Draw a straight line through the points.

EXAMPLE 1 *I sketch the graph of* $y = 2x + 3$.

X	−2	−1	0	1
y	−1	1	3	5



Example

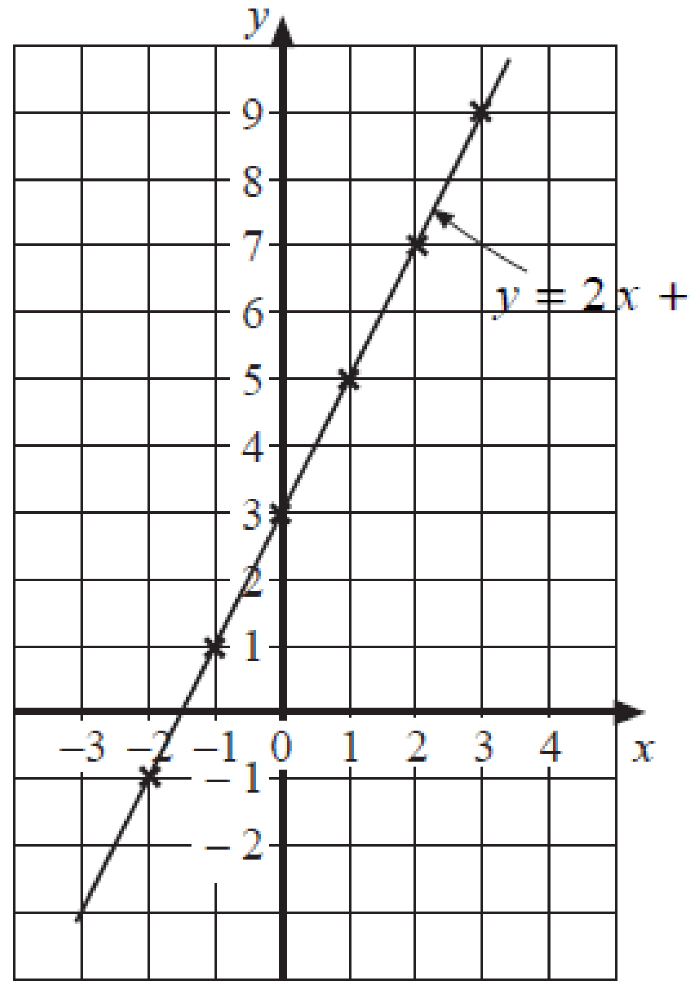
Draw the graph with equation $y = 2x + 3$.

Solution

First, find the coordinates of some points on the graph. This can be done by calculating y for a range of x values as shown in the table.

x	-2	-1	0	1	2	3
y	-1	1	3	5	7	9

The points can then be plotted on a set of axes and a straight line drawn through them.



Remark *An equation can be graphed by finding the intercepts, plotting the points and drawing a straight line through the intercepts.*

EXAMPLE 1 *I sketch the graph of $y = 2x + 3$.*

x-intercept

y=0 ➔ 0=2x+3

x= 2/3

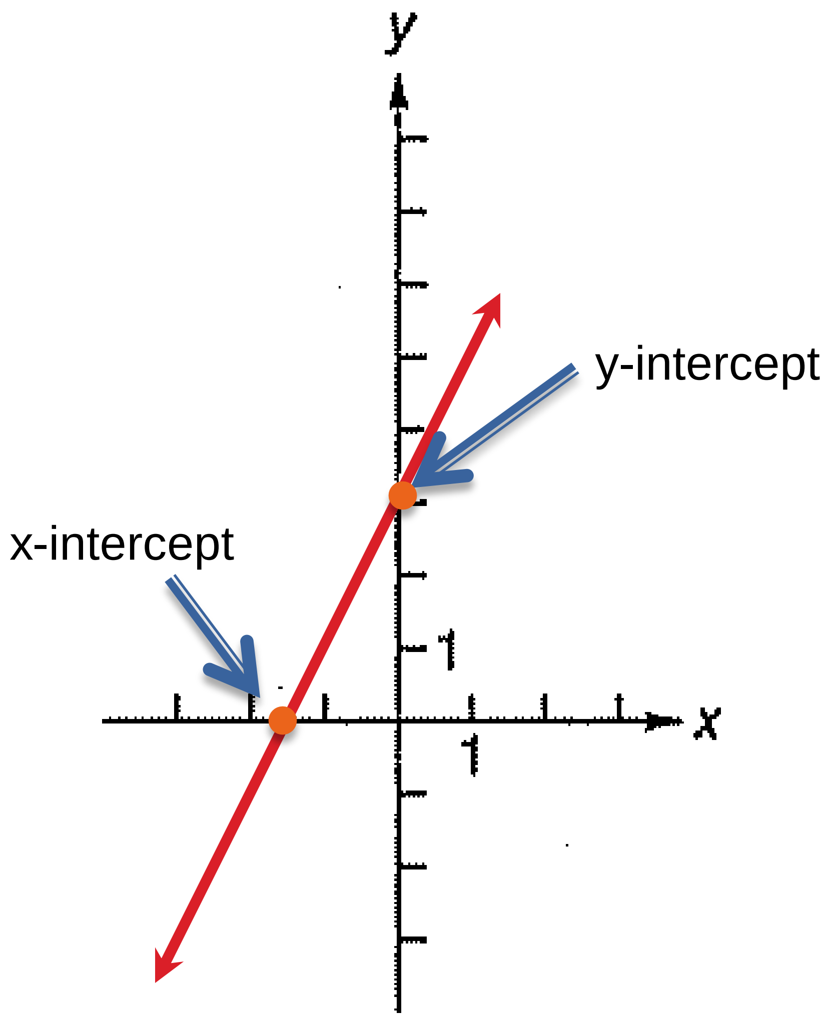
(2/3 , 0) is x-intercept

y- intercept

x=0 ➔ y=2.0+3

y= 3

(0,3) y-intercept



Example *Graph* $2x - 3y = 6.$

Solution *To find the x intercept we set $y = 0$ and solve the equation for x:*

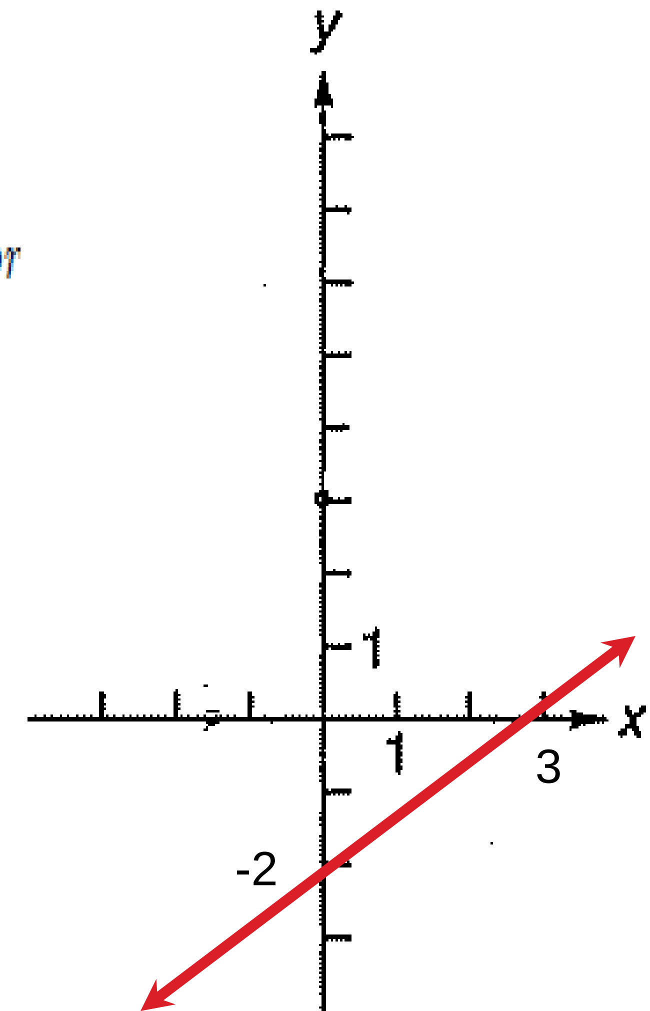
$$\begin{aligned} 2x - 3y &= 6, \\ 2x &= 6, \\ x &= 3. \end{aligned}$$

Hence, the x intercepts is $(3, 0)$.

To find the y intercept we set $x = 0$ and solve the equation for

$$\begin{aligned} 2x - 3y &= 6, \\ -3y &= 6, \\ y &= -2, \end{aligned}$$

which implies the y intercept is $(0, -2)$.



LINES

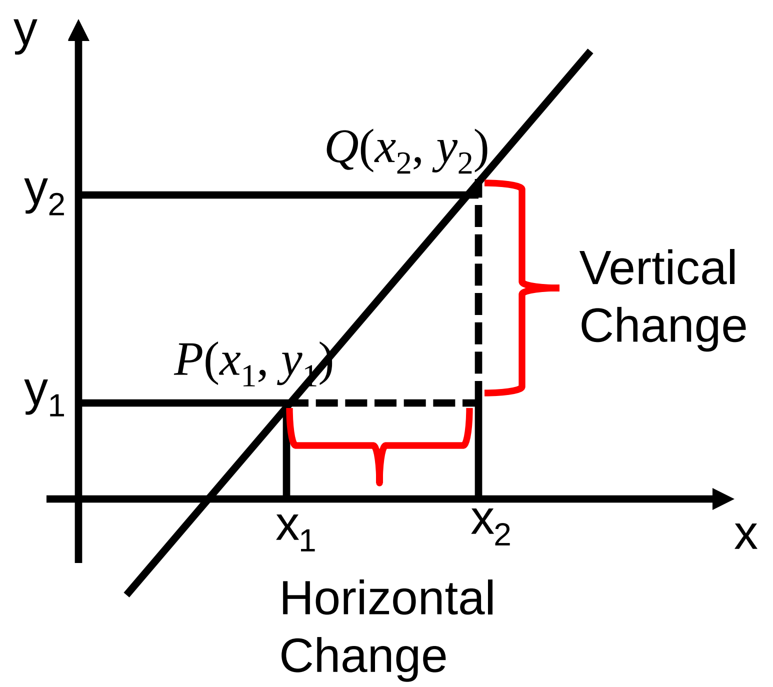
Slope of a Line

Many relationships between quantities can be represent conveniently by straight lines.

The slope of a nonvertical line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is denoted by m and is defined by

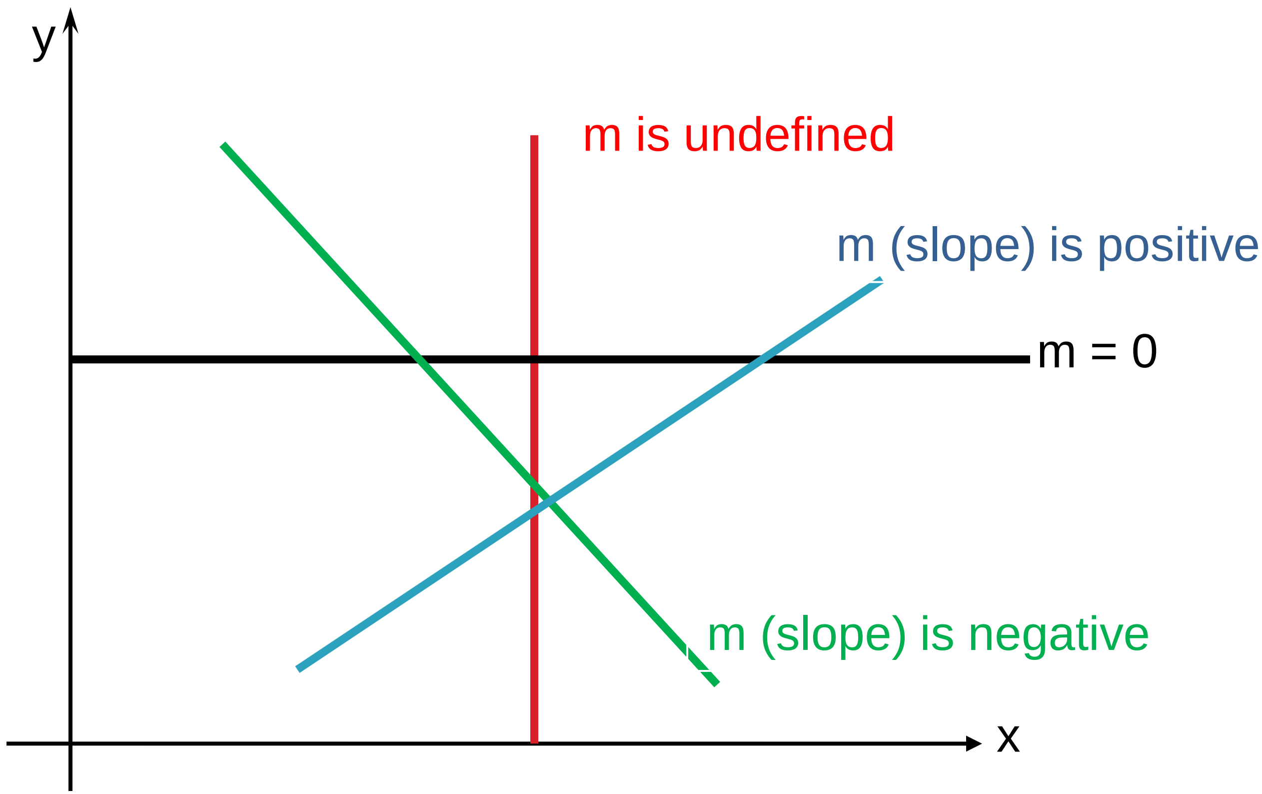
$$m = \frac{\text{vertical change}}{\text{Horizontal change}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}.$$



Vertical and Horizontal Lines

Either the "rise" or "run" could be zero



We can characterize the orientation of a line by its slope.

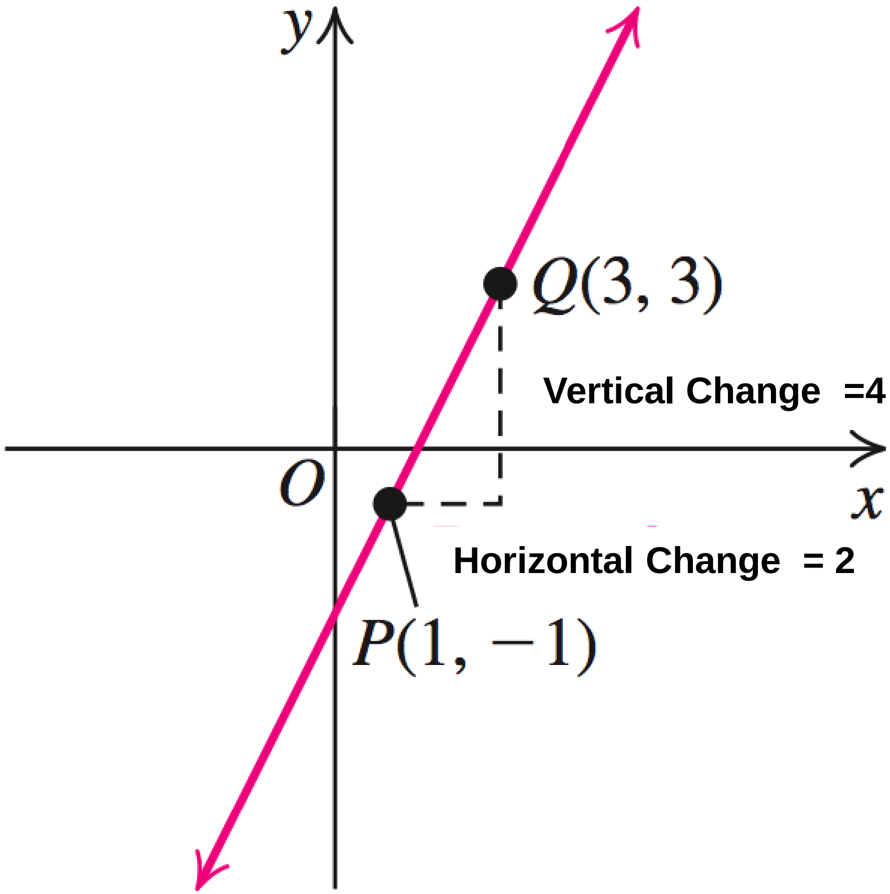
Zero Slope	horizontal line
Undefined slope	vertical line
Positive slope	line rises from left to right
Negative slope	line falls from left to right

EXAMPLE 1 Finding and Interpreting the Slope of a Line

Sketch the graph of the line that passes through the points *P*(1, –1) and *Q*(3, 3). Find and interpret the slope of the line.

Solution

Any two points determine a line; the graph of the line passing through the points *P*(1, –1) and *Q*(3, 3) is sketched here.



EXAMPLE 1 Finding and Interpreting the Slope of a Line

Solution continued

$P(1, -1)$ and $Q(3, 3)$

$$m = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{(3) - (-1)}{3 - (1)} = \frac{3+1}{3-1} = \frac{4}{2} = 2$$

Interpretation

The slope of this line is 2; this means that the value of *y* increases by exactly 2 units for every *increase* of 1 unit in the value of *x*. The graph is a straight line rising by 2 units for every one unit we go to the right.

EXAMPLE 2 Finding and Interpreting the Slope of a Line

The line in Figure 3.4 shows the relationship between the price p of a widget (in dollars) and the quantity q of widgets (in thousands) that consumers will buy at that price. Find and interpret the slope.

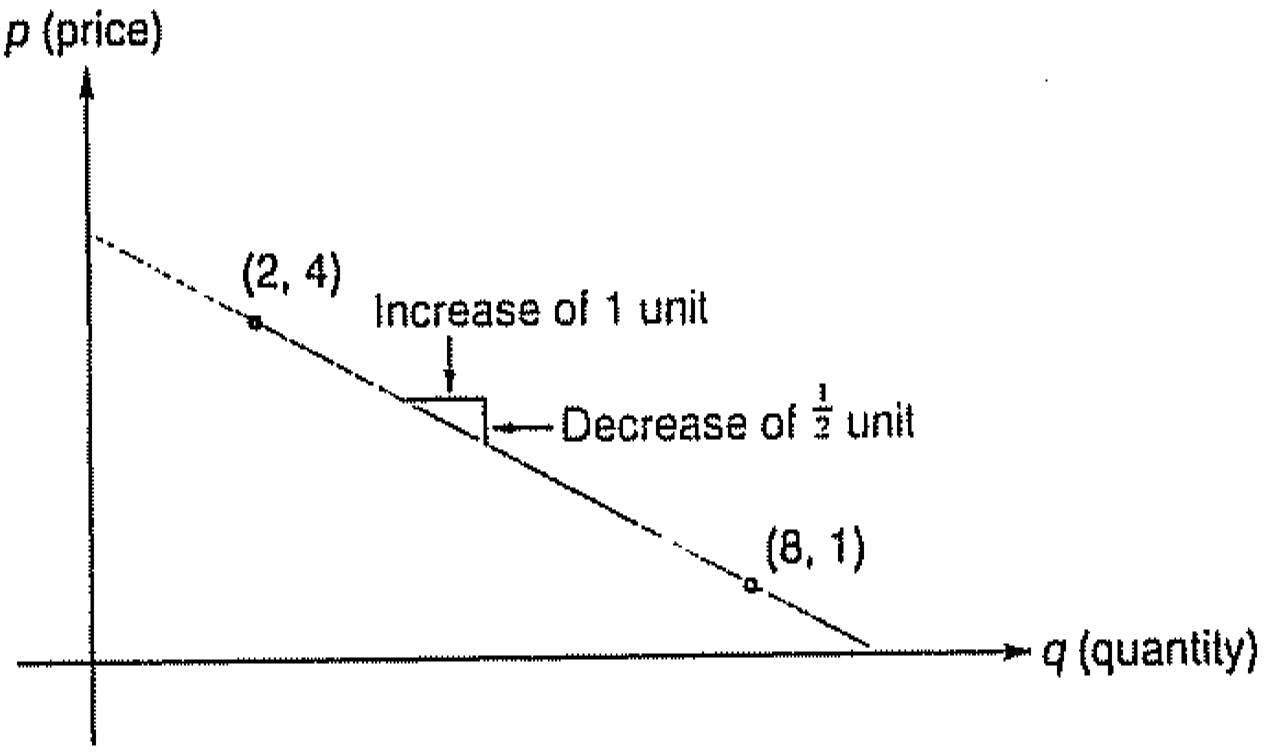


FIGURE 3.4 Price-quantity line.

Solution: In the slope formula (1), we replace the x's by q's and the y's by p's. Either point in Figure 3.4 may be chosen as (q_1, p_1). Letting $(2, 4) = (q_1, p_1)$ and $(8, 1) = (q_2, p_2)$, we have

$$m = \frac{p_2 - p_1}{q_2 - q_1} = \frac{1 - 4}{8 - 2} = \frac{-3}{6} = -\frac{1}{2}$$

The slope is negative, $-\frac{1}{2}$. This means that, for each 1-unit increase in quantity (one thousand widgets), there corresponds a **decrease** in price of $\frac{1}{2}$ (dollar per widget). Because of this decrease, the line **falls** from left to right.

Equations of Lines

Point-Slope Form: To find the equation of a line, when you only have two points. The point-slope form of the equation of a line is

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is a given point.

EXAMPLE 1

Finding an Equation of a Line with Given Point and Slope

Find the point–slope form of the equation of the line passing through the point (1, –2) and with slope m = 3. Then solve for y.

Solution

We have $x_1 = 1$, $y_1 = -2$, and $m = 3$.

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = 3(x - 1)$$

$$y + 2 = 3x - 3$$

$$y = 3x - 5$$

EXAMPLE 2 Finding an Equation of a Line with Given Point and Slope

Find the equation of a line that passes through the point (1,-3) with slope of 2

Solution: Using a point-slope form with $m = 2$ and $(x_1, y_1) = (1, -3)$ gives

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = 2(x - 1)$$

$$y + 3 = 2x - 2$$

which can be rewritten as

$$2x - y - 5 = 0$$

EXAMPLE 3 Finding an Equation of a Line Passing Through Two Given Points

Find the point–slope form of the equation of the line *l* passing through the points (–2, 1) and (3, 7). Then solve for *y*.

Solution

First, find the slope. $m = \frac{7-1}{3-(-2)} = \frac{6}{3+2} = \frac{6}{5}$

We have $x_1 = 3$, $y_1 = 7$.

$$y - y_1 = m(x - x_1)$$

$$y - 7 = \frac{6}{5}(x - 3)$$

$$y - 7 = \frac{6}{5}x - \frac{18}{5}$$

$$y = \frac{6}{5}x + \frac{17}{5}$$

EXAMPLE 4 | Finding an Equation of a Line with two Given Points

Find the equation of the line through the points (-5,7) and (4,16).

Solution:

$$m = \frac{16-7}{4-(-5)} = \frac{9}{9} = 1$$

Now use the point-slope form with m = 1 and (x_1, x_2) = (4,16). (We could just as well have used (-5,7)).

$$y - 16 = 1(x-4)$$

$$y = x - 4 + 16 = x + 12$$

EXAMPLE 5 Finding an Equation of a Line with a Given Slope and *y*-intercept

Find the point–slope form of the equation of the line with slope m and y-intercept b. Then solve for y.

Solution

The line passes through $(0, b)$.

$$y - y_1 = m(x - x_1)$$

$$y - b = m(x - 0)$$

$$y - b = mx$$

$$y = mx + b$$

SLOPE–INTERCEPT FORM OF THE EQUATION OF A LINE

The **slope-intercept form** of the equation of the line with slope m and y-intercept b is

$$y = mx + b.$$

EXAMPLE 1 | Slope – Intercept Form

Find an equation of the line with slope 3 and y-intercept −4.

Solution: Using the slope–intercept form $y = mx + b$ with $m = 3$ and $b = -4$ gives

$$y = 3x + (-4)$$

$$y = 3x - 4$$

EXAMPLE 2 Find the Slope and *y*-intercept of a line

Find the slope and y-intercept of the line with equation $y = 5(3 - 2x)$.

Solution:

Strategy: We shall rewrite the equation so it has the slope–intercept form $y = mx + b$. Then the slope is the coefficient of x and the y-intercept is the constant term.

We have

$$y = 5(3 - 2x)$$

$$y = 15 - 10x$$

$$y = -10x + 15$$

Thus, $m = -10$ and $b = 15$, so the slope is -10 and the y-intercept is 15.

If a *vertical* line passes through (a, b) (see Figure 3.8), then any other point (x, y) lies on the line if and only if $x = a$. The y-coordinate can have any value. Hence, an equation of the line is $x = a$. Similarly, an equation of the *horizontal* line passing through (a, b) is $y = b$. (See Figure 3.9.) Here the x-coordinate can have any value.

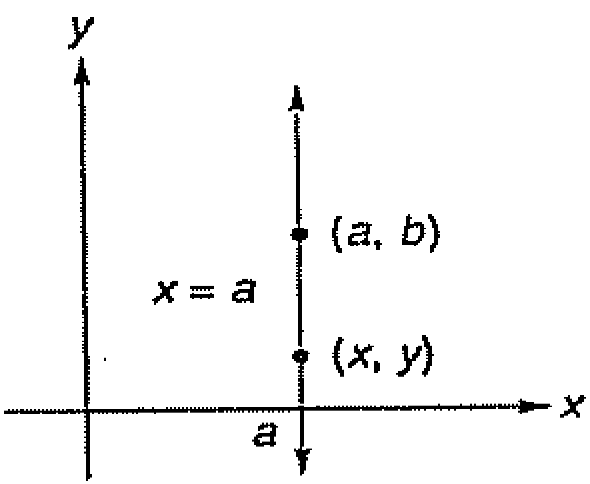


FIGURE 3.8 Vertical line through (a, b).

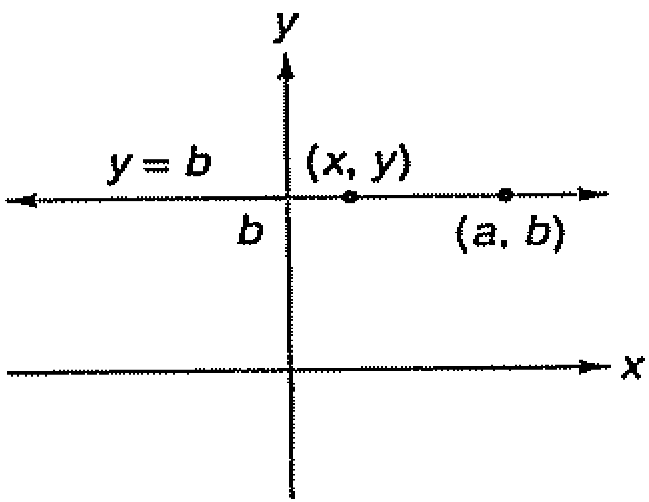


FIGURE 3.9 Horizontal line through (a, b).

EXAMPLE | Equations of Horizontal and Vertical Lines

a. An equation of the vertical line through $(-2, 3)$ is $x = -2$. An equation of the horizontal line through $(-2, 3)$ is $y = 3$.

TABLE 3.1 Forms of Equations of Straight Lines

Point–slope form	$y - y_1 = m(x - x_1)$
Slope–intercept form	$y = mx + b$
General linear form	$Ax + By + C = 0$
Vertical line	$x = a$
Horizontal line	$y = b$

PARALLEL AND PERPENDICULAR LINES

Let l_1 and l_2 be two distinct lines with slopes m_1 and m_2,respectively. Then

l_1 is parallel to l_2 if and only if $m_1 = m_2$.

l_1 is perpendicular l_2 to if and only if $m_1 \cdot m_2 = -1$.

Any two vertical lines are parallel, and any horizontal line is perpendicular to any vertical line.

EXAMPLE 2 Find the equation of the line passes through (3,-1) and (-2,-9).

Solution:

The slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-9+1}{-2-3} = \frac{8}{5}$$

The equation of this line is

$$y - y_1 = m\left(x - x_1\right)$$

$$y + 1 = \frac{8}{5}\left(x - 3\right)$$

$$y = \frac{8}{5}x - \frac{24}{5} - 1$$

$$y = \frac{8}{5}x - \frac{29}{5} \quad \text{(slope intercept form)}$$

The general form of this equation is $8x - 5y - 29 = 0$

EXAMPLE 3

Find the equation of the line passes through (-3,2) and parallel to y=4x-5.

Solution:

l_1 and l_2 are parallel $l_1 // l_2$, $m_1 = m_2$ $l_1:\ y = 4x - 5,\ m_1 = 4$

$$m_1 = m_2 = 4$$

The equation of the line l_2 is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 4(x + 3)$$

$$y = 4x + 12 + 2$$

$$y = 4x + 14 \quad (l_2)$$

EXAMPLE 4

Find the equation of the line passes through (-3,2) and perpendicular to y=4x-5.

l_1 is perpendicular to l_2,

$$m_1.m_2 = -1,\ 4m_2 = -1,\ m_2 = -\frac{1}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{1}{4}(x + 3)$$

$$y = -\frac{1}{4}x + \frac{5}{4}$$

EXAMPLE 5

Find the equation of the line passes through (2,-3) and it is vertical.

Solution:

Line passes through (2,-3) and it has no slope and y-interceptand it is vertical.

The equation of the line is $x = 2.$

EXAMPLE 6

Find the equation of the line passes through (7,4) and perpendicular to y=-4.

Solution:

Line passes through (7,4) and it is perpendicular to y=-4.

The slope of this line is $m_1 = 0$. $m_1.m_2 = -1,\ m_2 = -\frac{1}{0} = undefined$

The equation of the line l_2 is x=7.