3.11 Method of Least Squares

Suppose we have a set of observations $x_1, x_2, \ldots, x_n$. The sum of the squares of their deviations from some mean value is

$$S = \sum_{i=1}^{n} (x_i - x_m)^2 \quad \textbf{[3.22]}$$

Now, suppose we wish to minimize S with respect to the mean value x_m. We set

$$\frac{\partial S}{\partial x_m} = 0 = \sum_{i=1}^{n} -2(x_i - x_m) = -2\left(\sum_{i=1}^{n} x_i - nx_m\right) \quad \textbf{[3.23]}$$

where n is the number of observations. We find that

$$x_m = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \textbf{[3.24]}$$

or the mean value which minimizes the sum of the squares of the deviations is the arithmetic mean. This example might be called the simplest application of the method of least squares.

Suppose that the two variables x and y are measured over a range of values. Suppose further that we wish to obtain a simple analytical expression for y as a function of x. The simplest type of function is a linear one; hence, we might try to establish y as a linear function of x. (Both x and y may be complicated functions of other parameters so arranged that x and y vary approximately in a linear manner. This matter will be discussed later.) The problem is one of finding the *best* linear function, for the data may scatter a considerable amount. We could solve the problem rather quickly by plotting the data points on graph paper and drawing a straight line through them by eye. Indeed this is common practice, but the method of least squares gives a more reliable way to obtain a better functional relationship than the guesswork of plotting. We seek an equation of the form

$$y = ax + b \quad \textbf{[3.25]}$$

We therefore wish to minimize the quantity

$$S = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2 \quad \textbf{[3.26]}$$

This is accomplished by setting the derivatives with respect to a and b equal to zero.

Performing these operations, there results

$$nb + a\sum x_i = \sum y_i \tag{3.27}$$

$$b\sum x_i + a\sum x_i^2 = \sum x_i y_i \tag{3.28}$$

Solving Eqs. (3.27) and (3.28) simultaneously gives

$$a = \frac{n\sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{n\sum x_i^2 - \left(\sum x_i\right)^2} \tag{3.29}$$

$$b = \frac{\left(\sum y_i\right)\left(\sum x_i^2\right) - \left(\sum x_i y_i\right)\left(\sum x_i\right)}{n\sum x_i^2 - \left(\sum x_i\right)^2} \tag{3.30}$$

Designating the computed value of y as $\hat{y}$, we have

$$\hat{y} = ax + b$$

and the standard error of estimate of y for the data is

$$\text{Standard error} = \left[\frac{\sum (y_i - \hat{y}_i)^2}{n-2}\right]^{1/2} \tag{3.31}$$

$$= \left[\frac{\sum (y_i - ax_i - b)^2}{n-2}\right]^{1/2} \tag{3.32}$$

The method of least squares may also be used for determining higher-order polynomials for fitting data. One only needs to perform additional differentiations to determine additional constants. For example, if it were desired to obtain a least-squares fit according to the quadratic function

$$y = ax^2 + bx + c$$

the quantity

$$S = \sum_{i=1}^{n} \left[y_i - (ax_i^2 + bx_i + c)\right]^2$$

would be minimized by setting the following derivatives equal to zero:

$$\frac{\partial S}{\partial a} = 0 = \sum 2\left[y_i - (ax_i^2 + bx_i + c)\right](-x_i^2)$$

$$\frac{\partial S}{\partial b} = 0 = \sum 2\left[y_i - (ax_i^2 + bx_i + c)\right](-x_i)$$

$$\frac{\partial S}{\partial c} = 0 = \sum 2\left[y_i - (ax_i^2 + bx_i + c)\right](-1)$$

Expanding and collecting terms, we have

$$a\sum x_i^4 + b\sum x_i^3 + c\sum x_i^2 = \sum x_i^2 y_i \tag{3.33}$$

$$a\sum x_i^3 + b\sum x_i^2 + c\sum x_i = \sum x_i y_i \tag{3.34}$$

$$a\sum x_i^2 + b\sum x_i + cn = \sum y_i \qquad \textbf{[3.35]}$$

These equations may then be solved for the constants a, b, and c.

Regression Analysis

In the above discussion of the method of least squares no mention has been made of the influence of experimental uncertainty on the calculation. We are considering the method primarily for its utility in fitting an algebraic relationship to a set of data points. Clearly, the various x_i and y_i could have different experimental uncertainties. To take all these into account requires a rather tedious calculation procedure, which we shall not present here; however, we may state the following rules:

1. If the values of x_i and y_i are taken as the data value in y and the value of x *on the fitted curve for the same value of y*, then there is a presumption that the uncertainty in x is large compared with that in y.
2. If the values of x_i and y_i are taken as the data value in y and the value *on the fitted curve for the same value of x*, the presumption is that the uncertainty in y dominates.
3. If the uncertainties in x_i and y_i are believed to be of approximately equal magnitude, a special averaging technique must be used.

In rule 1 we say we are taking a *regression* of x on y, and in rule 2 there is a regression of y on x. In the second case we are minimizing the sum of the squares of the deviations of the actual points from the assumed curve and also assuming *that x does not vary appreciably at each point*. If we obtained

$$y = a + bx$$

and then solved to get

$$x = \frac{1}{b}y - \frac{a}{b}$$

this second relation would *not* necessarily give a good calculation for x since the minimization was carried out in the y direction and not in the x direction. In Example 3.19 rule 2 is assumed to apply.

Example 3.19 **LEAST-SQUARES REGRESSION.** From the following data obtain y as a linear function of x using the method of least squares:

y_i	x_i
1.2	1.0
2.0	1.6
2.4	3.4
3.5	4.0
3.5	5.2
$\sum y_i = 12.6$	$\sum x_i = 15.2$

Solution

We seek an equation of the form

$$y = ax + b$$

We first calculate the quantities indicated in the following table:

$x_i y_i$	x_i^2
1.2	1.0
3.2	2.56
8.16	11.56
14.0	16.0
18.2	27.04
$\sum x_i y_i = 44.76$	$\sum x_i^2 = 58.16$

We calculate the value of a and b using Eqs. (3.29) and (3.30) with $n = 5$:

$$a = \frac{(5)(44.76) - (15.2)(12.6)}{(5)(58.16) - (15.2)^2} = 0.540$$

$$b = \frac{(12.6)(58.16) - (44.76)(15.2)}{(5)(58.16) - (15.2)^2} = 0.879$$

Thus, the desired relation is

$$y = 0.540x + 0.879$$

A plot of this relation and the data points from which it was derived is shown in the accompanying figure.

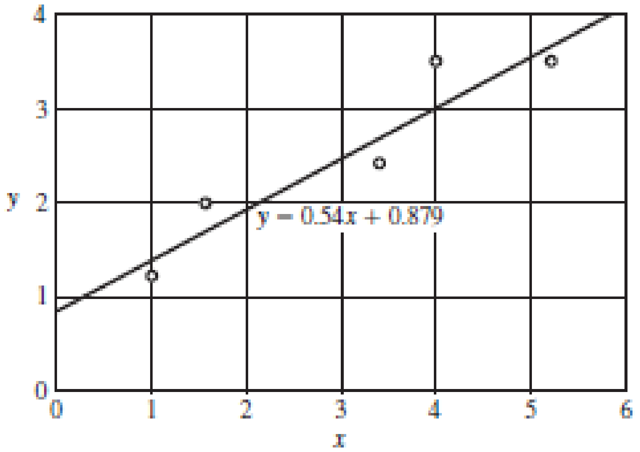

Figure Example 3.19

3.12 The Correlation Coefficient

Let us assume that a suitable correlation between y and x has been obtained, by either least-squares analysis or graphical curve fitting. We want to know how good this fit is and the parameter which conveys this information is the *correlation coefficient* r defined by

$$r = \left[1 - \frac{\sigma_{y,x}^2}{\sigma_y^2}\right]^{1/2} \qquad \textbf{[3.36]}$$

where σ_y is the standard deviation of y given as

$$\sigma_y = \left[\frac{\sum_{i=1}^{n}(y_i - y_m)^2}{n-1}\right]^{1/2} \qquad \textbf{[3.37]}$$

and

$$\sigma_{y,x} = \left[\frac{\sum_{i=1}^{n}(y_i - y_{ic})^2}{n-2}\right]^{1/2} \qquad \textbf{[3.38]}$$

The y_i are the actual values of y, and the y_{ic} are the values computed from the correlation equation for the same value of x.

The division by $n - 2$ results from the fact that we have used the two derived variables a and b in determining the value of y_{ic}. We might say that this removes 2 degrees of freedom from the system of data. The correlation coefficient r may also be written as

$$r^2 = \frac{\sigma_y^2 - \sigma_{y,x}^2}{\sigma_y^2} \qquad \textbf{[3.39]}$$

where, now, r^2 is called the *coefficient of determination*. We note that for a perfect fit $\sigma_{y,x} = 0$ because there are no deviations between the data and the correlation. In this case $r = 1.0$. If $\sigma_y = \sigma_{y,x}$, we obtain $r = 0$, indicating a poor fit or substantial scatter around the fitted line. The reader must be cautioned about ascribing too much virtue to values of r close to 1.0. These values may occur when the data do *not* fit the line.

To be on the safe side, one should *never* accept a least-squares analysis based *only* on calculations. One should *always* plot the data to obtain a visual observation of the behavior. If the data points do indeed hug the least-squares line, then a high value of r will be indicative of a very good correlation. If the data scatter but still appear to follow the fitted relationship, then a small value of r will also be meaningful as a measure of poorer correlation.

At this point we must stress the need for graphical displays of data for other purposes. In our discussions of uncertainties and errors we noted that the experimentalist may be the best person to assess the uncertainties in the primary measurements. Sometimes during the course of the experiment the experimenter may note that a particular data point has an erratic behavior and so record the observation in the lab notebook. When the data are plotted, such a point may be excluded if it appears out of line with other data or retained if it appears satisfactory. If a least-squares analysis was performed which included all the data, the correlation might not be so good as could be obtained with exclusion of the questionable data point(s). For these reasons seasoned experimentalists like to get an "eyeball" plot of the correlating straight line before actually performing a least-squares analysis. In Sec. 3.15 we shall see how one may go about obtaining straight-line plots for different functional relationships.

It may be noted that most scientific calculators have built-in routines which calculate the correlation coefficient as well as other statistical functions. In addition, there are many computer software packages which accomplish these calculations, for example, those of Refs. [15], [16], and [28].

A relationship for the correlation coefficient which may be preferable to Eq. (3.36) for computer calculations is

$$r = \frac{n\sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{\left[n\sum x_i^2 - \left(\sum x_i\right)^2\right]^{1/2}\left[n\sum y_i^2 - \left(\sum y_i\right)^2\right]^{1/2}} \qquad \textbf{[3.40]}$$

In Eqs. (3.33) through (3.35) we noted the technique that one might apply for a least-squares fit to a quadratic function. In this case the correlation coefficient is still given by Eq. (3.36), but now

$$\sigma_{y,x} = \left[\frac{(y_i - y_{ic})^2}{n-3}\right]^{1/2} \qquad \textbf{[3.41]}$$

In general, for fit with a polynomial of order m one would obtain

$$\sigma_{y,x} = \left[\frac{(y_i - y_{ic})^2}{n-(m+1)}\right]^{1/2} \qquad \textbf{[3.42]}$$

Aside from the fact that one may anticipate the form of the functional data relationship from theory, we may sometimes try to fit the data with a polynomial through a least-squares analysis. For those who are computer inclined the tendency is to assume that the higher the order of the polynomial, the better the correlation will be. As a result, a certain overkill may be experienced. In some cases a higher-order polynomial may actually provide a poorer correlation than the simple quadratic.

3.13 MULTIVARIABLE REGRESSION

The least-squares method may be extended to perform a regression analysis for more than one variable. In the linear case we would have the form

$$y = b + m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \qquad \textbf{[3.43]}$$

where the x_n are the independent variables. For only two variables we form the sum of the squares

$$S = \sum (y_i - b - m_1 x_{1,i} - m_2 x_{2,i})^2 \qquad \textbf{[3.44]}$$

and minimize this sum with the differentiations:

$$\frac{\partial S}{\partial b} = -2 \sum (y_i - b - m_1 x_{1,i} - m_2 x_{2,i}) = 0$$

$$\frac{\partial S}{\partial m_1} = -2 \sum x_{1,i}(y_i - b - m_1 x_{1,i} - m_2 x_{2,i}) = 0$$

$$\frac{\partial S}{\partial m_2} = -2 \sum x_{2,i}(y_i - b - m_1 x_{1,i} - m_2 x_{2,i}) = 0$$

This set of linear equations may then be solved for the coefficients m_1, m_2, and b.

A further extension of the multivariable regression method may be made to an exponential form where

$$y = b m_1^{x1} m_2^{x2} \cdots m_n^{xn} \qquad \textbf{[3.45]}$$

The equation may be modified by taking the logarithm of each side to give

$$\log y = \log b + x_1 \log m_1 + x_2 \log m_2 + \cdots \qquad \textbf{[3.46]}$$

A least-squares analysis is then performed to determine values of the constants b, m_1, m_2, etc. The calculation of correlation coefficients for multivariable regressions is described in Ref. [12].

Computer software packages are available to perform the multivariable calculations indicated above. Microsoft Excel is one such package, and examples of its use in multivariable linear and exponential regression analysis, including computation of correlation coefficients, are given in Ref. [28]. The standard deviation for multivariable regression is computed with Eq. (3.42) with the term $(m + 1)$ replaced by the number of constants to be determined. For the two-variable linear relation in Eq. (3.44) there are three constants to be determined so the denominator of Eq. (3.42) would be $n - 3$.

Multivariable regression calculations can become rather involved and are best performed with a computer. Some words of caution are in order though. It is very easy

CORRELATION COEFFICIENT. Calculate the correlation coefficient for the least-square correlation of Example 3.19.

Example 3.20

Solution

From Example 3.19

$$y_m = \frac{\sum y_i}{n} = \frac{12.6}{5} = 2.52$$

and from the correlating equation $y_{ic} = 0.5490x + 0.879$:

i	y_i	y_{ic}	$(y_i - y_{ic})^2$
1	1.2	1.419	0.048
2	2.0	1.743	0.066
3	2.4	2.715	0.0992
4	3.5	3.039	0.212
5	3.5	3.687	0.035
			$\sum = 0.4607$

so that

$$\sigma_{y,x} = \left(\frac{0.4607}{3}\right)^{1/2} = 0.3919$$

In addition, we have

$$y_m = (\Sigma y_i)/5 = 2.52$$
$$\sigma_y = [\Sigma(y_i - y_m)^2/(5-1)]^{1/2} = 0.987$$

so that the correlation coefficient is

$$r = \left[1 - \left(\frac{0.3919}{0.987}\right)^2\right]^{1/2} = 0.9178$$

Regression, Residual, and Total Sum of Squares

Let us define the following terms:

$$S_{\text{total}} = \Sigma(y_i - y_m)^2$$
$$S_{\text{regression}} = \Sigma(y_{ic} - y_m)^2$$
$$S_{\text{residual}} = \Sigma(y_i - y_{ic})^2$$

Because

$$(y_i - y_m) = (y_i - y_{ic}) + (y_{ic} - y_m)$$

it follows that

$$S_{\text{total}} = S_{\text{residual}} + S_{\text{regression}}$$

A perfect "goodness of fit" will be obtained when $S_{\text{residual}} = 0$, that is, the regression curve passes through the data points. The measure of goodness of fit may then be taken as

$$r^2 = S_{\text{regression}}/S_{\text{total}} = (S_{\text{total}} - S_{\text{residual}})/S_{\text{total}}$$

3.14 Standard Deviation of the Mean

We have taken the arithmetic mean value as the best estimate of the true value of a set of experimental measurements. Considerable discussion has been devoted to data subjected to random uncertainties and to an examination of the various types of errors and deviations that may occur in an experimental measurement. But one very important question has not yet been answered: How *good* (or precise) is this arithmetic mean value which is taken as the best estimate of the true value of a set of readings? To obtain an experimental answer to this question it would be necessary to repeat the set of measurements and to find a new arithmetic mean. In general, we would find that this new arithmetic mean would differ from the previous value, and thus we would not be able to resolve the problem until a large number of *sets of data* was collected. We would then know how well the mean of a single set approximated the mean which would be obtained with a large number of sets. The mean value of a large number of sets is presumably the true value. Consequently, we wish to know the standard deviation of the mean of a single set of data from this true value.

It turns out that the problem may be resolved with a statistical analysis which we shall not present here. The result is

$$\sigma_m = \frac{\sigma}{\sqrt{n}} \qquad \textbf{[3.47]}$$

where σ_m = standard deviation of the mean value

σ = standard deviation of the set of measurements

n = number of measurements in the set

We should note that the calculation of statistical parameters like standard deviation and least-square fits to data is easily performed with standard computer programs which are available on even small hand calculators.

Example 3.21

UNCERTAINTY IN MEAN VALUE. For the data of Example 3.7, estimate the uncertainty in the calculated mean value of the readings.

Solution

We shall make this estimate for the original data and for the reduced data of Example 3.13. For the original data the standard deviation of the mean is

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{0.627}{\sqrt{10}} = 0.198 \text{ cm}$$

The arithmetic mean value calculated in Example 3.7 was $x_m = 5.613$ cm. We could now specify the uncertainty of this value by using the odds of Table 3.3:

$$\begin{aligned} x_m &= 5.613 \pm 0.198 \text{ cm} \qquad (2.15 \text{ to } 1) \\ &= 5.756 \pm 0.396 \text{ cm} \qquad (21 \text{ to } 1) \\ &= 5.613 \pm 0.594 \text{ cm} \qquad (369 \text{ to } 1) \end{aligned}$$

Using the data of Example 3.13, where one point has been eliminated by Chauvenet's criterion, we may make a better estimate of the mean value with less uncertainty. The standard deviation of the mean is calculated as

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{0.465}{\sqrt{9}} = 0.155 \text{ cm}$$

for the mean value of 5.756 cm. Thus, we would estimate the uncertainty as

$$\begin{aligned} x_m &= 5.756 \pm 0.155 \text{ cm} \qquad (2.15 \text{ to } 1) \\ &= 5.756 \pm 0.310 \text{ cm} \qquad (21 \text{ to } 1) \\ &= 5.756 \pm 0.465 \text{ cm} \qquad (369 \text{ to } 1) \end{aligned}$$

3.15 STUDENT'S *t*-DISTRIBUTION

In Sec. 3.14 we have used the relation

$$\sigma_m = \frac{\sigma}{\sqrt{n}}$$

to determine the standard deviation of the mean in terms of the standard deviation of the population. For small samples ($n < 10$) this relation has been shown to be

Table 3.7 Values of Student's t for use in Equation (3.48)

Subscript designates percent confidence level.

Degrees of freedom ν	t_{50}	t_{80}	t_{90}	t_{95}	t_{98}	t_{99}	$t_{99.9}$
1	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	0.816	1.886	2.920	4.303	6.965	9.925	31.598
3	0.765	1.638	2.353	3.182	4.541	5.841	12.941
4	0.741	1.533	2.132	2.776	3.747	4.604	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	6.859
6	0.718	1.440	1.943	2.447	3.143	3.707	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	5.405
8	0.706	1.397	1.860	2.306	2.896	3.355	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.767
24	0.685	1.318	1.711	2.064	2.492	2.797	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.725
26	0.684	1.315	1.706	2.056	2.479	2.779	3.707
27	0.684	1.314	1.703	2.052	2.473	2.771	3.690
28	0.683	1.313	1.701	2.048	2.467	2.763	3.674
29	0.683	1.311	1.699	2.045	2.462	2.756	3.659
30	0.683	1.310	1.697	2.042	2.457	2.750	3.646
40	0.681	1.303	1.684	2.021	2.423	2.704	3.551
60	0.679	1.296	1.671	2.000	2.390	2.660	3.460
120	0.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.291

somewhat unreliable. A better method for estimating confidence intervals was developed by Student[2] by introducing the variable t such that

$$\Delta = \frac{t\sigma}{\sqrt{n}} \tag{3.48}$$

where, now, t replaces the z variable previously used. It can be shown that

$$t = \frac{\bar{x} - X}{\sigma}\sqrt{n} \tag{3.49}$$

where n = number of observations
$\bar{x}$ = mean of n observations
X = mean of normal population which the samples are taken from

Student then developed a distribution function $f(t)$ such that

$$f(t) = \frac{K_0}{\left(1 + \frac{t^2}{n - i}\right)^{n/2}}$$

$$f(t) = K_0\left(1 + \frac{t^2}{v}\right)(v + 1)/2 \tag{3.50}$$

where K_0 is a constant which depends on n and v is $(n - 1)$ degrees of freedom. When $n \to \infty$, the distribution function approaches the normal distribution. Table 3.7 gives values of Student's t for different degrees of freedom and levels of confidence. t_{90} means a 90 percent confidence level. Note that for $v \to \infty$, t_{90} is 1.645, which agrees with Table 3.4.

Example 3.22

CONFIDENCE LEVEL FROM t-DISTRIBUTION. Ten observations of a voltage are made with $\bar{x} = 15$ V and $\sigma = \pm 0.1$ V. Determine the 5 and 1 percent significance levels.

Solution

For $n = 10$ we have $v = 10 - 1 = 9$. At the 5 percent significance level the probability is 95 percent and we find from Table 3.7

$$t = 2.262$$

From Eq. (3.48)

$$\Delta = \frac{(2.262)(0.1)}{\sqrt{10}} = 0.0715 \text{ V}$$

at the 1 percent significance level $P = 0.99$. $v = 9$, and

$$t = 3.250$$

[2] Pen name of William S. Gosset (1876–1937), an Irish chemist.

so that

$$\Delta = \frac{(3.25)(0.1)}{10} = 0.1028 \text{ V}$$

We thus could state with a 95 percent confidence level that the voltage is 15 V ± 0.0715 V or with a 99 percent confidence level that it is 15 V $\pm$ 0.1028 V.

Example 3.23 **ESTIMATE OF SAMPLE SIZE.** For the steel bar in Example 3.11, obtain a new estimate for the number of measurements required using the t-distribution.

Solution

From Example 3.10 we have

$$\Delta = 0.2 \text{ mm}$$
$$\sigma = 0.5 \text{ mm}$$

and from Eq. (3.44)

$$\Delta = \frac{t\sigma}{\sqrt{n}}$$

or

$$t = 0.4\sqrt{n} \qquad \textbf{[a]}$$

At this point we note that t is a function of n through Table 3.7 so that Eq. (a) must be solved by iteration to obtain a value for n. Remembering that $\nu = n - 1 = 24$, the trials are for the 95 percent confidence level:

n	t_{95} (from Table 3.7)	t [Calculated, Eq. (a)]
25	2.064	2.000
26	2.060	2.040
27	2.056	2.078

Therefore, we shall require 27 measurements for the t-distribution in contrast to the 25 measurements required in Example 3.11.

Example 3.24 **CONFIDENCE LEVEL.** Ten measurements are made of the thickness of a metal plate which give 3.61, 3.62, 3.60, 3.63, 3.61, 3.62, 3.60, 3.62, 3.64, and 3.62 mm. Determine the mean value and the tolerance limits for a 90 percent confidence level.

Solution

The mean value is calculated from

$$x_m = \frac{1}{n}\sum^{i} x_i = \frac{1}{10}(36.17) = 3.617 \text{ mm}$$

The sample standard deviation is calculated from

$$\sigma = \left[\frac{\sum(x_i - x_m)^2}{n-1}\right]^{1/2}$$

$$= \left[\frac{1.41 \times 10^{-3}}{10-1}\right]^{1/2} = 0.0125 \text{ mm}$$

Entering Table 3.7 with $\nu = 10 - 1 = 9$, we obtain t_{90} for a 90 percent confidence level:

$$t_{90} = 1.833$$

Thus, we have from Eq. (3.44)

$$\Delta = \frac{t\sigma}{10} = \frac{(1.833)(0.0125)}{10}$$
$$= 0.00726 \text{ mm}$$

or $$x = 3.617 \text{ mm} \pm 0.00726 \text{ mm} \quad \text{(90 percent confidence)}$$

Example 3.25

CONFIDENCE LEVEL. If the results of the measurements of Example 3.24 are stated as

$$x_m = 3.617 \text{ mm} \pm 0.01 \text{ mm}$$

what confidence level should be assigned to this statement?

Solution

We still have $\sigma = 0.0125$ mm and from Eq. (3.44)

$$\Delta = \frac{t\sigma}{\sqrt{n}} = 0.01 = \frac{t(0.0125)}{10}$$

or $$t = 2.53$$

Entering Table 3.7 with $\nu = 10 - 1 = 9$, we find by interpolation

$$t = 2.53 = t_{96.4}$$

indicating a confidence level of 96.4 percent.

Example 3.26

LOWER CONFIDENCE LEVEL. Repeat Examples 3.24 and 3.25 for a confidence level of 90 percent (10 percent level of significance).

Solution

We still have

$$\Delta = 0.2 \text{ mm}$$
$$\sigma = 0.5 \text{ mm}$$

For 90 percent confidence level we obtain from Table 3.7

$$z = 1.65$$

and from

$$\Delta = \frac{z\sigma}{\sqrt{n}} = 0.2 = \frac{(1.65)(0.5)}{\sqrt{n}}$$

$$n = 17.01, \quad \text{rounded to } n = 18$$

For the t-distribution

$$\Delta = \frac{t\sigma}{\sqrt{n}}$$

and

$$t = 0.4\sqrt{n} \qquad \textbf{[a]}$$

and, again, an iterative procedure is required. This time we must use t_{90} from Table 3.7. The trials are, with $\nu = n - 1$:

n	t_{90} (from Table 3.7)	t [Calculated, Eq. (a)]
17	1.746	1.649
18	1.740	1.697
19	1.734	1.743

and 19 measurements would be required by the t-distribution.