MENG353 - FLUID MECHANICS

SOURCE: FUNDAMENTALS OF FLUIDMECHANICS
MUNSON,P.GERHART,A.GERHART and HOCHSTEIN

CHAPTER 2 FLUID STATICS

FALL 2017 - 18

ASSOC.PROF.DR.HASAN HACIŞEVKİ

EASTERN MEDITERRANEAN UNIVERSITY

Learning Objectives

After completing this chapter, you should be able to:

- determine the pressure at various locations in a fluid at rest.
- explain the concept of manometers and apply appropriate equations to determine pressures.
- calculate the hydrostatic pressure force on a plane or curved submerged surface.
- calculate the buoyant force and determine the stability of floating or submerged objects.

2.1 Pressure at a Point

As we briefly discussed in Chapter 1, the term *pressure* is used to indicate the normal force per unit area at a given point acting on a given plane within the fluid mass of interest. A question that immediately arises is how the pressure at a point varies with the orientation of the plane passing through the point.

Although we are primarily interested in fluids at rest, to make the analysis as general as possible, we will allow the fluid element to have accelerated motion. The assumption of zero shearing stresses will still be valid as long as the fluid mass moves as a rigid body; that is, there is no relative motion between adjacent elements.

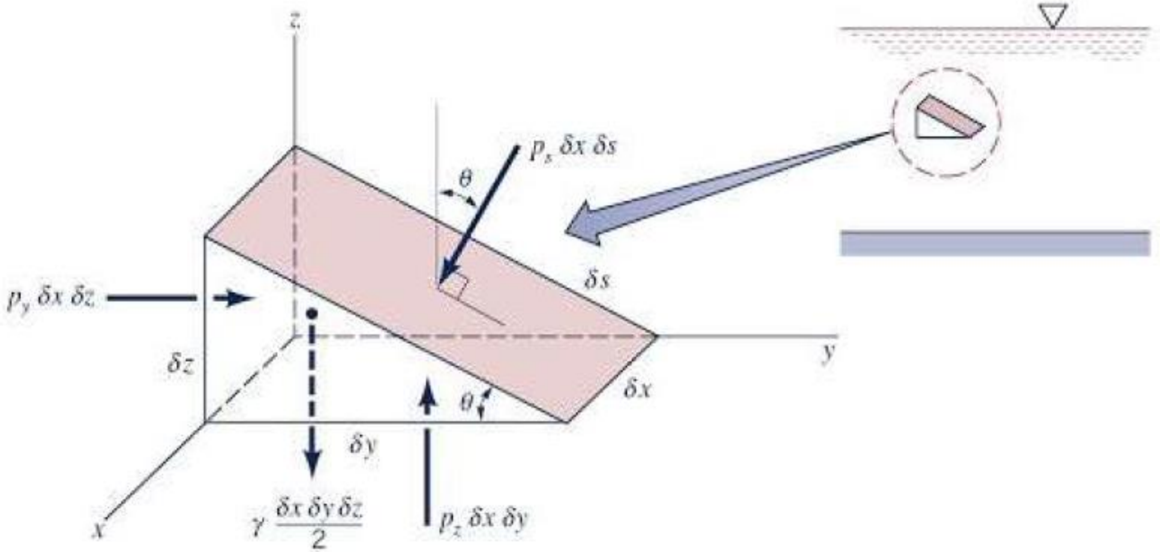


Figure 2.1 **Forces on an arbitrary wedge-shaped element of fluid.**

The equations of motion (Newton's second law, $\mathbf{F} = m\mathbf{a}$) in the y and z directions are, respectively,

$$\sum F_y = p_y\, \delta x\, \delta z - p_s\, \delta x\, \delta s \sin\theta = \rho \frac{\delta x\, \delta y\, \delta z}{2} a_y$$

$$\sum F_z = p_z\, \delta x\, \delta y - p_s\, \delta x\, \delta s \cos\theta - \gamma \frac{\delta x\, \delta y\, \delta z}{2} = \rho \frac{\delta x\, \delta y\, \delta z}{2} a_z$$

$$\delta y = \delta s \cos\theta \qquad \delta z = \delta s \sin\theta$$

so that the equations of motion can be rewritten as

$$p_y - p_s = \rho a_y \frac{\delta y}{2}$$

$$p_z - p_s = (\rho a_z + \gamma)\frac{\delta z}{2}$$

Since we are really interested in what is happening at a point, we take the limit as δx, δy, and δz approach zero (while maintaining the angle θ), and it follows that

$$p_y = p_s \qquad p_z = p_s$$

or $p_s = p_y = p_z$. The angle θ was arbitrarily chosen so we can conclude that *the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present*. This important result is known as ***Pascal's law,*** named in honor of Blaise Pascal (1623–1662), a French mathematician who made important contributions in the field of hydrostatics.

2.2 Basic Equation for Pressure Field

Although we have answered the question of how the pressure at a point varies with direction, we are now faced with an equally important question—how does the pressure in a fluid in which there are no shearing stresses vary from point to point? To answer this question, consider a small rectangular

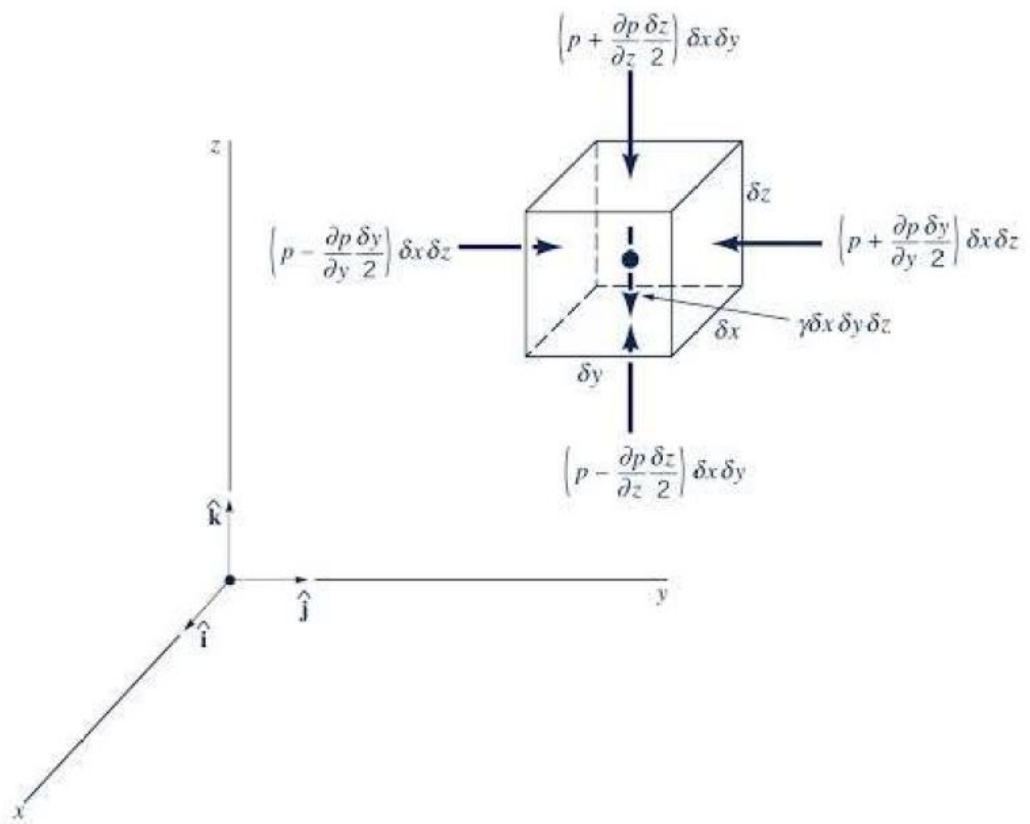
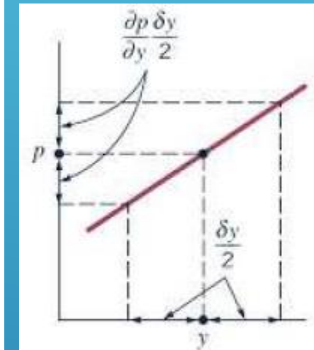


Figure 2.2 **Surface and body forces acting on small fluid element.**



element of fluid removed from some arbitrary position within the mass of fluid of interest as illustrated in Fig. 2.2. There are two types of forces acting on this element: ***surface forces*** due to the pressure and a ***body force*** equal to the weight of the element. Other possible types of body forces, such as those due to magnetic fields, will not be considered in this text.

If we let the pressure at the center of the element be designated as p, then the average pressure on the various faces can be expressed in terms of p and its derivatives, as shown in Fig. 2.2. We are actually using a Taylor series expansion of the pressure at the element center to approximate the pressures a short distance away and neglecting higher order terms that will vanish as we let δx, δy, and δz approach zero. This is illustrated by the figure in the margin. For simplicity the surface forces in the x direction are not shown. The resultant surface force in the y direction is

$$\delta F_y = \left(p - \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z - \left(p + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z$$

or

$$\delta F_y = -\frac{\partial p}{\partial y}\,\delta x\,\delta y\,\delta z$$

Similarly, for the x and z directions the resultant surface forces are

$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z \qquad \delta F_z = -\frac{\partial p}{\partial z}\,\delta x\,\delta y\,\delta z$$

The resultant surface force acting on the element can be expressed in vector form as

$$\delta \mathbf{F}_s = \delta F_x \hat{\mathbf{i}} + \delta F_y \hat{\mathbf{j}} + \delta F_z \hat{\mathbf{k}}$$

or

$$\delta \mathbf{F}_s = -\left(\frac{\partial p}{\partial x} \hat{\mathbf{i}} + \frac{\partial p}{\partial y} \hat{\mathbf{j}} + \frac{\partial p}{\partial z} \hat{\mathbf{k}} \right) \delta x \, \delta y \, \delta z \qquad \textbf{(2.1)}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors along the coordinate axes shown in Fig. 2.2. The group of terms in parentheses in Eq. 2.1 represents in vector form the *pressure gradient* and can be written as

$$\frac{\partial p}{\partial x} \hat{\mathbf{i}} + \frac{\partial p}{\partial y} \hat{\mathbf{j}} + \frac{\partial p}{\partial z} \hat{\mathbf{k}} = \nabla p$$

where

$$\nabla(\ ) = \frac{\partial(\ )}{\partial x}\hat{\mathbf{i}} + \frac{\partial(\ )}{\partial y}\hat{\mathbf{j}} + \frac{\partial(\ )}{\partial z}\hat{\mathbf{k}}$$

and the symbol ∇ is the *gradient* or "del" vector operator. Thus, the resultant surface force per unit volume can be expressed as

$$\frac{\delta \mathbf{F}_s}{\delta x\, \delta y\, \delta z} = -\nabla p$$

Since the z axis is vertical, the weight of the element is

$$-\delta \mathcal{W}\hat{\mathbf{k}} = -\gamma\, \delta x\, \delta y\, \delta z\, \hat{\mathbf{k}}$$

where the negative sign indicates that the force due to the weight is downward (in the negative z direction). Newton's second law, applied to the fluid element, can be expressed as

$$\sum \delta \mathbf{F} = \delta m\, \mathbf{a}$$

where $\Sigma\, \delta \mathbf{F}$ represents the resultant force acting on the element, $\mathbf{a}$ is the acceleration of the element, and δm is the element mass, which can be written as $\rho\, \delta x\, \delta y\, \delta z$. It follows that

$$\sum \delta \mathbf{F} = \delta \mathbf{F}_s - \delta \mathcal{W}\hat{\mathbf{k}} = \delta m\, \mathbf{a}$$

or

$$-\nabla p\, \delta x\, \delta y\, \delta z - \gamma\, \delta x\, \delta y\, \delta z\, \hat{\mathbf{k}} = \rho\, \delta x\, \delta y\, \delta z\, \mathbf{a}$$

and, therefore,

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \mathbf{a} \qquad \textbf{(2.2)}$$

2.3 Pressure Variation in a Fluid at Rest

For a fluid at rest $\mathbf{a} = 0$ and Eq. 2.2 reduces to

$$\nabla p + \gamma \hat{\mathbf{k}} = 0$$

or in component form

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\gamma \tag{2.3}$$

These equations show that for the coordinates defined, pressure is not a function of x or y. Thus, as we move from point to point in a horizontal plane (any plane parallel to the x–y plane), the pressure does not change. Since p depends only on z, the last of Eqs. 2.3 can be written as the ordinary differential equation

$$\boxed{\frac{dp}{dz} = -\gamma} \tag{2.4}$$

Equation 2.4 is the fundamental equation for fluids at rest and can be used to determine how pressure changes with elevation. This equation and the figure in the margin indicate that the pressure gradient in the vertical direction is negative; that is, the pressure decreases as we move upward in a fluid at rest. There is no requirement that γ be a constant. Thus, it is valid for fluids with constant specific weight, such as liquids, as well as fluids whose specific weight may vary with elevation,

2.3.1 Incompressible Fluid

Since the specific weight is equal to the product of fluid density and acceleration due to gravity ($\gamma = \rho g$), changes in γ are caused by a change in either ρ or g. For most engineering applications the variation in g is negligible, so our main concern is with the possible variation in the fluid density. In general, a fluid with constant density is called an ***incompressible fluid.*** For liquids the variation in density is usually negligible, even for large changes in pressure, so that the assumption of constant specific weight when dealing with liquids is usually a good one. For this instance, Eq. 2.4 can be directly integrated

$$\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz$$

to yield

$$p_2 - p_1 = -\gamma(z_2 - z_1)$$

or

$$p_1 - p_2 = \gamma(z_2 - z_1) \tag{2.5}$$

where p_1 and p_2 are pressures at the vertical elevations z_1 and z_2, as is illustrated in Fig. 2.3.

Equation 2.5 can be written in the compact form

$$p_1 - p_2 = \gamma h \tag{2.6}$$

or

$$p_1 = \gamma h + p_2 \tag{2.7}$$

where h is the distance, $z_2 - z_1$, which is the depth of fluid measured downward from the location of p_2. This type of pressure distribution is commonly called a ***hydrostatic distribution,*** and Eq. 2.7 shows that in an incompressible fluid at rest the pressure varies linearly with depth. The pressure must increase with depth to "hold up" the fluid above it.

EXAMPLE 2.1 Pressure–Depth Relationship

GIVEN Because of a leak in a buried gasoline storage tank, water has seeped in to the depth shown in Fig. E2.1. The specific gravity of the gasoline is $SG = 0.68$.

FIND Determine the pressure at the gasoline–water interface and at the bottom of the tank. Express the pressure in units of lb/ft^2, $lb/in.^2$, and as a pressure head in feet of water.

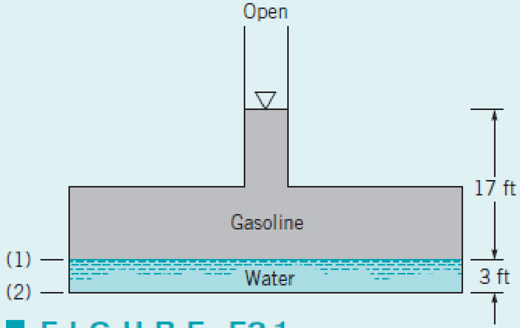


FIGURE E2.1

SOLUTION

Since we are dealing with liquids at rest, the pressure distribution will be hydrostatic, and therefore the pressure variation can be found from the equation:

$$p = \gamma h + p_0$$

With p_0 corresponding to the pressure at the free surface of the gasoline, then the pressure at the interface is

$$\begin{aligned} p_1 &= SG\gamma_{H_2O}h + p_0 \\ &= (0.68)(62.4 \text{ lb/ft}^3)(17 \text{ ft}) + p_0 \\ &= 721 + p_0 \ (\text{lb/ft}^2) \end{aligned}$$

If we measure the pressure relative to atmospheric pressure (gage pressure), it follows that $p_0 = 0$, and therefore

$$p_1 = 721 \text{ lb/ft}^2 \quad \textbf{(Ans)}$$

$$p_1 = \frac{721 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 5.01 \text{ lb/in.}^2 \quad \textbf{(Ans)}$$

$$\frac{p_1}{\gamma_{H_2O}} = \frac{721 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = 11.6 \text{ ft} \quad \textbf{(Ans)}$$

It is noted that a rectangular column of water 11.6 ft tall and 1 ft^2 in cross section weighs 721 lb. A similar column with a 1-$in.^2$ cross section weighs 5.01 lb.

We can now apply the same relationship to determine the pressure at the tank bottom; that is,

$$\begin{aligned} p_2 &= \gamma_{H_2O} h_{H_2O} + p_1 \\ &= (62.4 \text{ lb/ft}^3)(3 \text{ ft}) + 721 \text{ lb/ft}^2 \quad \textbf{(Ans)} \\ &= 908 \text{ lb/ft}^2 \end{aligned}$$

$$p_2 = \frac{908 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 6.31 \text{ lb/in.}^2 \quad \textbf{(Ans)}$$

$$\frac{p_2}{\gamma_{H_2O}} = \frac{908 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = 14.6 \text{ ft} \quad \textbf{(Ans)}$$

COMMENT Observe that if we wish to express these pressures in terms of *absolute* pressure, we would have to add the local atmospheric pressure (in appropriate units) to the previous results. A further discussion of gage and absolute pressure is given in Section 2.5.

2.3.2 Compressible Fluid

We normally think of gases such as air, oxygen, and nitrogen as being **compressible fluids** because the density of the gas can change significantly with modest changes in pressure and temperature. Thus, although Eq. 2.4 applies at a point in a gas, it is necessary to consider the possible variation in γ before the equation can be integrated.

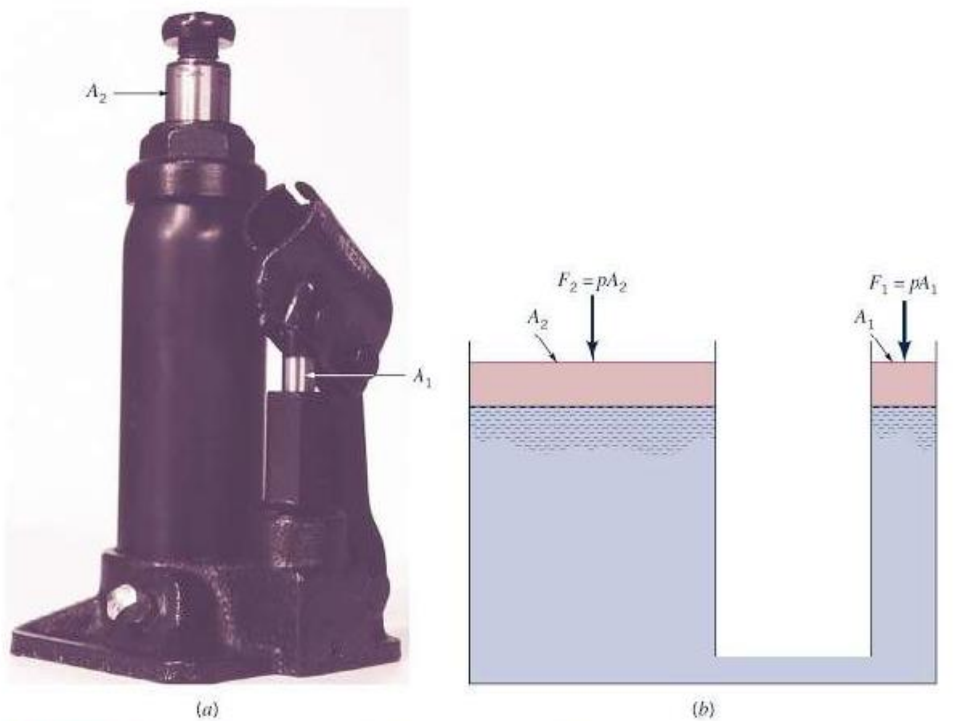


Figure 2.5 **(*a*) Hydraulic jack, (*b*) Transmission of fluid pressure.**

For those situations in which the variations in heights are large, on the order of thousands of feet, attention must be given to the variation in the specific weight. As is described in Chapter 1, the equation of state for an ideal (or perfect) gas is

$$p = \rho R T$$

where p is the absolute pressure, R is the gas constant, and T is the absolute temperature. This relationship can be combined with Eq. 2.4 to give

$$\frac{dp}{dz} = -\frac{gp}{RT}$$

and by separating variables

$$\int_{p_1}^{p_2} \frac{dp}{p} = \ln \frac{p_2}{p_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{T} \qquad \textbf{(2.9)}$$

where g and R are assumed to be constant over the elevation change from z_1 to z_2. Although the acceleration of gravity, g, does vary with elevation, the variation is very small (see Tables C.1 and C.2 in Appendix C), and g is usually assumed constant at some average value for the range of elevation involved.

Before completing the integration, one must specify the nature of the variation of temperature with elevation. For example, if we assume that the temperature has a constant value T_0 over the range z_1 to z_2 (*isothermal* conditions), it then follows from Eq. 2.9 that

$$p_2 = p_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \qquad \textbf{(2.10)}$$

2.4 Standard Atmosphere

An important application of Eq. 2.9 relates to the variation in pressure in the earth's atmosphere. Ideally, we would like to have measurements of pressure versus altitude over the specific range for the specific conditions (temperature, reference pressure) for which the pressure is to be determined.

Since the temperature variation is represented by a series of linear segments, it is possible to integrate Eq. 2.9 to obtain the corresponding pressure variation. For example, in the troposphere, which extends to an altitude of about 11 km (~36,000 ft), the temperature variation is of the form

$$T = T_a - \beta z \tag{2.11}$$

TABLE 2.1

Properties of U.S. Standard Atmosphere at Sea Level[a]

Property	SI Units	BG Units
Temperature, T	288.15 K (15 °C)	518.67 °R (59.00 °F)
Pressure, p	101.33 kPa (abs)	2116.2 lb/ft^2 (abs) [14.696 lb/in.^{2} (abs)]
Density, ρ	1.225 kg/m^3	0.002377 slugs/ft^3
Specific weight, γ	12.014 N/m^3	0.07647 lb/ft^3
Viscosity, μ	1.789×10^{-5} N · s/m^2	3.737×10^{-7} lb · s/ft^2

[a]Acceleration of gravity at sea level = 9.807 m/s^2 = 32.174 ft/s^2.

where T_a is the temperature at sea level ($z = 0$) and β is the *lapse rate* (the rate of change of temperature with elevation). For the standard atmosphere in the troposphere, $\beta = 0.00650$ K/m or 0.00357 °R/ft.

Equation 2.11 used with Eq. 2.9 yields

$$p = p_a\left(1 - \frac{\beta z}{T_a}\right)^{g/R\beta} \tag{2.12}$$

2.5 Measurement of Pressure

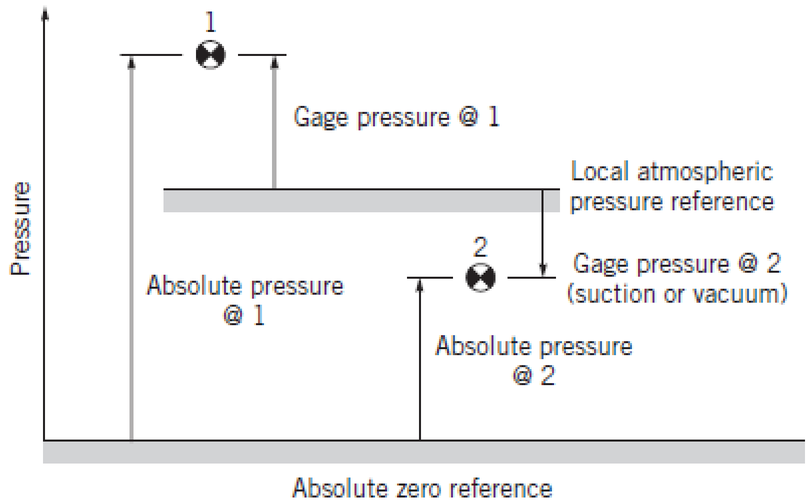


FIGURE 2.7 Graphical representation of gage and absolute pressure.

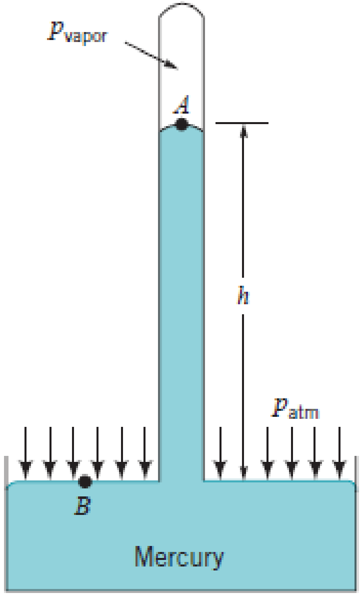


FIGURE 2.8 Mercury barometer.

The measurement of atmospheric pressure is usually accomplished with a mercury ***barometer***, which in its simplest form consists of a glass tube closed at one end with the open end immersed in a container of mercury as shown in Fig. 2.8. The tube is initially filled with mercury (inverted with its open end up) and then turned upside down (open end down), with the open end in the container of mercury. The column of mercury will come to an equilibrium position where its weight plus the force due to the vapor pressure (which develops in the space above the column) balances the force due to the atmospheric pressure. Thus,

$$p_{atm} = \gamma h + p_{vapor} \qquad \textbf{(2.13)}$$

EXAMPLE 2.3 Barometric Pressure

GIVEN A mountain lake has an average temperature of 10 °C and a maximum depth of 40 m. The barometric pressure is 598 mm Hg.

FIND Determine the absolute pressure (in pascals) at the deepest part of the lake.

SOLUTION

The pressure in the lake at any depth, h, is given by the equation

$$p = \gamma h + p_0$$

where p_0 is the pressure at the surface. Since we want the absolute pressure, p_0 will be the local barometric pressure expressed in a consistent system of units; that is

$$\frac{p_{\text{barometric}}}{\gamma_{\text{Hg}}} = 598 \text{ mm} = 0.598 \text{ m}$$

and for $\gamma_{\text{Hg}} = 133 \text{ kN/m}^3$

$$p_0 = (0.598 \text{ m})(133 \text{ kN/m}^3) = 79.5 \text{ kN/m}^2$$

From Table B.2, $\gamma_{\text{H}_2\text{O}} = 9.804 \text{ kN/m}^3$ at 10 °C and therefore

$$\begin{aligned} p &= (9.804 \text{ kN/m}^3)(40 \text{ m}) + 79.5 \text{ kN/m}^2 \\ &= 392 \text{ kN/m}^2 + 79.5 \text{ kN/m}^2 \\ &= 472 \text{ kPa (abs)} \end{aligned}$$

(Ans)

COMMENT This simple example illustrates the need for close attention to the units used in the calculation of pressure; that is, be sure to use a *consistent* unit system, and be careful not to add a pressure head (m) to a pressure (Pa).

2.6.1 Piezometer Tube

Since manometers involve columns of fluids at rest, the fundamental equation describing their use is Eq. 2.8

$$p = \gamma h + p_0$$

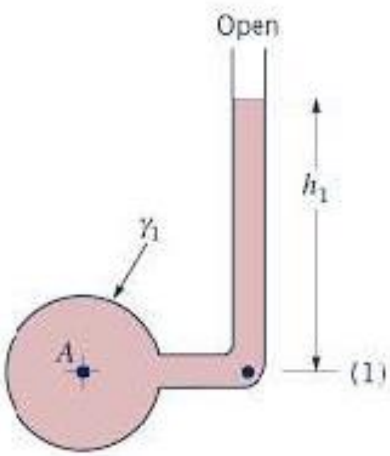
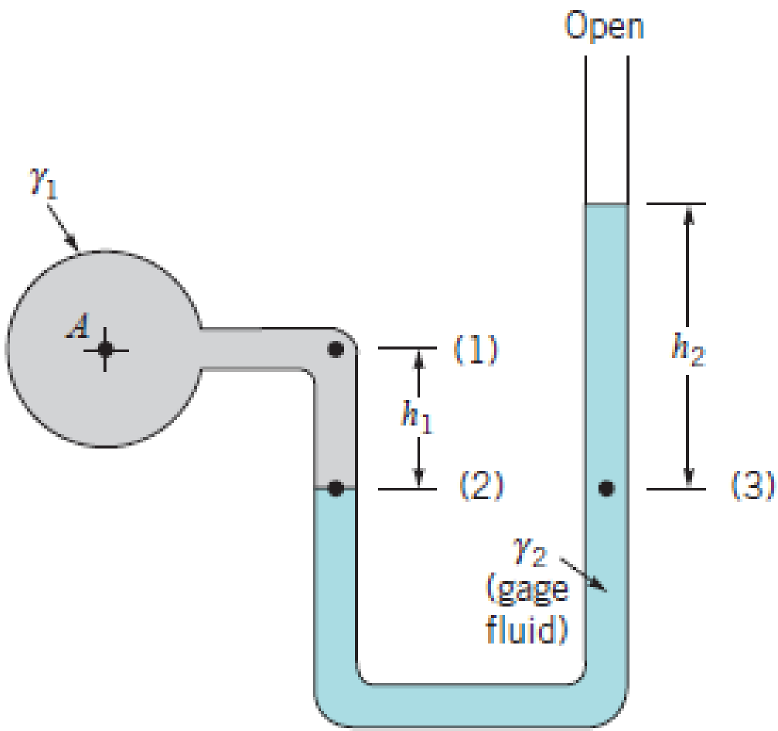


Figure 2.9 **Piezometer tube.**

2.6.2 U-Tube Manometer



■ FIGURE 2.10 Simple U-tube manometer.

Forming the manometer tube in a U-shape as shown in Fig. 2.10 provides the option of introducing a *gauge fluid* with a higher specific weight than the fluid in the container in which the pressure is to be measured. The first, most obvious, capability added by the introduction of the U-shape with a gauge fluid is the ability to measure the pressure in a gas. Again, the pressure to be measured, p_A, is related to the column heights. A numerical value can be computed by repeated application of Eq. 2.8. The following sequence of equations is generated by taking an imaginary "walk" though the manometer from p_A to the open end, and repeatedly applying Eq. 2.8.

step 1: $p_1 = p_A$ (same elevation)

step 2: $p_2 = p_1 + \gamma_1 h_1 = p_A + \gamma_1 h_1$

step 3: $p_3 = p_2$

step 4: $p_{atm} = p_3 + \gamma_2(-h_2) = p_A + \gamma_1 h_1 - \gamma_2 h_2$

or

$$p_A = p_{atm} - \gamma_1 h_1 + \gamma_2 h_2$$

This process can be codified into a simple **manometer rule** by recalling the implications of Eq. 2.8:

1. Write the pressure at either end of the manometer;
2. Proceed through the manometer, adding γh if moving to a greater depth or subtracting γh if moving to a lesser depth;
3. Stop at the far end, or any point in between, and set the expression equal to the local pressure.

If you start to apply the manometer rule at the "far" end of the manometer, the equation you seek is produced in a single line of work. For the manometer of Fig. 2.10:

$$p_{atm} + \gamma_2 h_2 - \gamma_1 h_1 = p_A \qquad \textbf{(2.14)}$$

Thinking about this result quickly reveals two useful results. First, if the pressure to be measured is in a gas, $\gamma_2 \gg \gamma_1$, so to a very good approximation: $p_A = p_{atm} + \gamma_2 h_2$, the same result as was obtained for the piezometer. Second, we can adjust the sensitivity, and therefore the resolution and range, of the pressure measurement device. Increasing the density of the gauge fluid allows practical measurement of much larger pressures but sacrifices resolution. Decreasing the density of the gauge fluid limits the practical range of pressure measurement but increases the resolution. The most common gauge fluids are water, mercury ($SG \approx 13.6$), and oil ($SG \approx 0.8$).

The U-tube manometer is also widely used to measure the difference in pressure between two containers or two points in a given system. Consider a manometer connected between containers A and B as is shown in Fig. 2.11. Application of the manometer rule produces an equation for the pressure difference between the containers. Starting at p_B:

$$p_B + \gamma_3 h_3 + \gamma_2 h_2 - \gamma_1 h_1 = p_A$$

or

$$p_A - p_B = \gamma_3 h_3 + \gamma_2 h_2 - \gamma_1 h_1.$$

Carefully include units when replacing symbols with information provided in a problem statement to help ensure a correct solution.

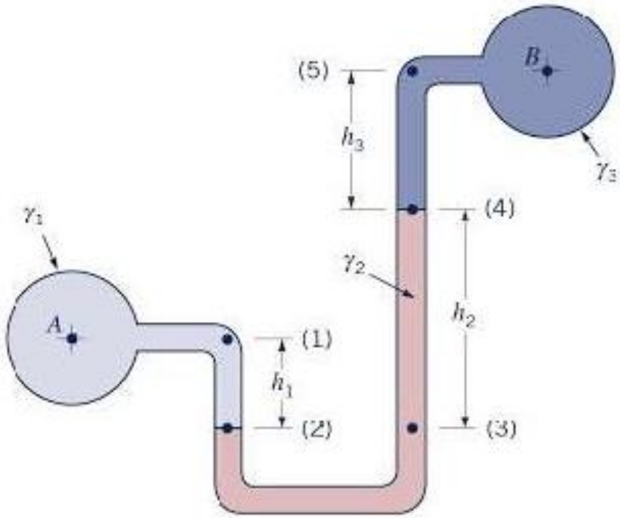


Figure 2.11 **Differential U-tube manometer.**

EXAMPLE 2.4 Simple U-Tube Manometer

GIVEN A closed tank contains compressed air and oil ($SG_{oil} = 0.90$) as is shown in Fig. E2.4. A U-tube manometer using mercury ($SG_{Hg} = 13.6$) is connected to the tank as shown. The column heights are $h_1 = 36$ in., $h_2 = 6$ in., and $h_3 = 9$ in.

FIND Determine the pressure reading (in psi) of the gage.

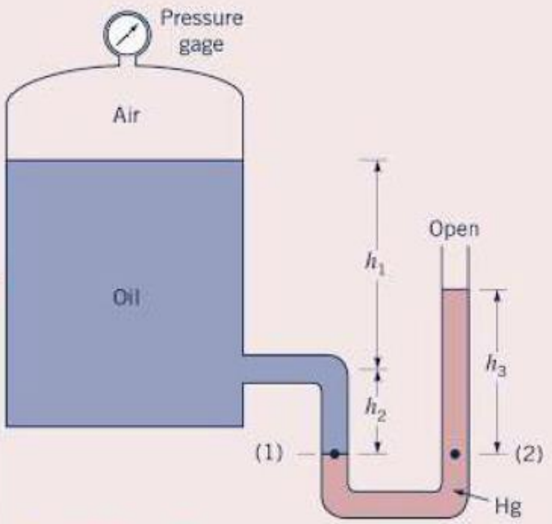


Figure E2.4

SOLUTION

Applying the manometer rule, starting at the open end of the manometer:

$$p_{atm} + \gamma_{Hg} h_3 - \gamma_{oil} h_2 - \gamma_{oil} h_1 = p_{Air}$$

The hydrostatic variation of pressure within the air has been neglected because its density is much smaller than that of either liquid. Using our convention that, unless otherwise stated, pressures will be gauge pressures:

$$p_{Air} = 0 + (SG_{Hg})(\gamma_{H_2O})h_3 + (SG_{oil})(\gamma_{H_2O})(-h_2 - h_1)$$

$$p_{Air} = \left(62.4 \frac{\text{lb}}{\text{ft}^3}\right)\left[(13.6)\left(\frac{9}{12}\text{ft}\right) - (0.9)\left(\frac{6 + 36}{12}\text{ft}\right)\right]$$

$$p_{Air} = 440 \frac{\text{lb}}{\text{ft}^2} \times \frac{1\ \text{ft}^2}{144\ \text{in}^2} = 3.06\ \text{psi} \qquad \textbf{(Ans)}$$

COMMENTS Note that the air pressure is a function of the height of the mercury in the manometer and the depth of the oil (both in the tank and in the tube). It is not just the mercury in the manometer that is important.

Assume that the gage pressure remains at 3.06 psi, but the manometer is altered so that it contains only oil. That is, the mercury is replaced by oil. A simple calculation shows that in this case the vertical oil-filled tube would need to be $h_3 = 11.3$ ft tall, rather than the original $h_3 = 9$ in. There is an obvious advantage of using a heavy fluid such as mercury in manometers.

EXAMPLE 2.5 U-Tube Manometer

GIVEN As will be discussed in Chapter 3, the volume rate of flow, Q, through a pipe can be determined by means of a flow nozzle located in the pipe as illustrated in Fig. E2.5*a*. The nozzle creates a pressure drop, $p_A - p_B$, along the pipe that is related to the flow through the equation $Q = K\sqrt{p_A - p_B}$, where K is a constant depending on the pipe and nozzle size. The pressure drop is frequently measured with a differential U-tube manometer of the type illustrated.

FIND **(a)** Determine an equation for $p_A - p_B$ in terms of the specific weight of the flowing fluid, γ_1, the specific weight of the gage fluid, γ_2, and the various heights indicated. **(b)** For $\gamma_1 = 9.80\ \text{kN/m}^3$, $\gamma_2 = 15.6\ \text{kN/m}^3$, $h_1 = 1.0\ \text{m}$, and $h_2 = 0.5\ \text{m}$, what is the value of the pressure drop, $p_A - p_B$?

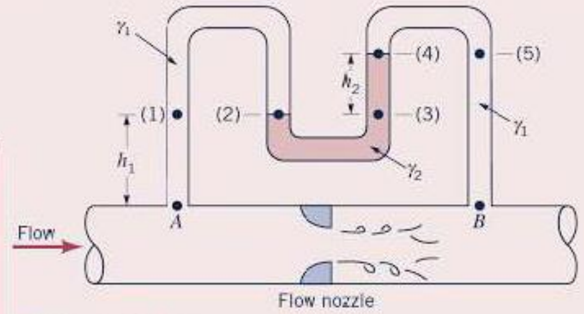


Figure E2.5*a*

SOLUTION

(a) Although the fluid in the pipe is moving, the fluids in the columns of the manometer are at rest so that the pressure variation in the manometer tubes is hydrostatic. Applying the manometer rule, starting at A:

$$p_A - \gamma_1 h_1 - \gamma_2 h_2 + \gamma_1(h_1 + h_2) = p_B$$

or

$$p_A - p_B = (\gamma_2 - \gamma_1)h_2 \qquad \textbf{(Ans)}$$

COMMENT It is to be noted that the only column height of importance is the differential reading, h_2. The differential manometer could be placed 0.5 or 5.0 m above the pipe ($h_1 = 0.5$ m or $h_1 = 5.0$ m), and the value of h_2 would remain the same.

(b) The specific value of the pressure drop for the data given is

$$p_A - p_B = (0.5\ \text{m})(15.6\ \text{kN/m}^3 - 9.80\ \text{kN/m}^3)$$
$$= 2.90\ \text{kPa} \qquad \textbf{(Ans)}$$

COMMENT By repeating the calculations for manometer fluids with different specific weights, γ_2, the results shown in Fig. E2.5*b* are obtained. Note that relatively small pressure differences can be measured if the manometer fluid has nearly the same specific weight as the flowing fluid. It is the difference in the specific weights, $\gamma_2 - \gamma_1$, that is important.

Hence, by rewriting the answer as $h_2 = (p_A - p_B)/(\gamma_2 - \gamma_1)$ it is seen that even if the value of $p_A - p_B$ is small, the value of h_2 can be large enough to provide an accurate reading provided the value of $\gamma_2 - \gamma_1$ is also small.

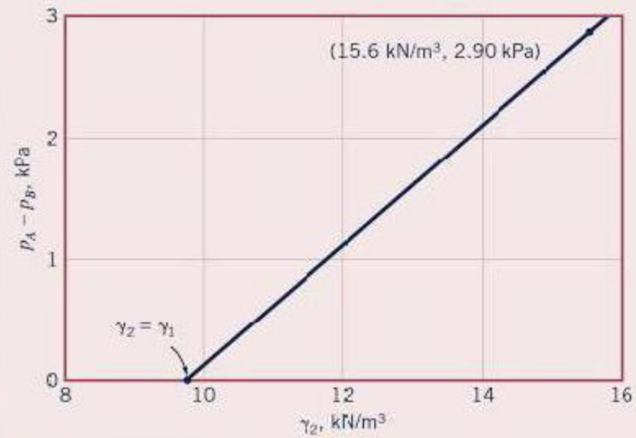


Figure E2.5*b*

2.6.3 Inclined-Tube Manometer

To measure small pressure changes, a manometer of the type shown in Fig. 2.12 is frequently used. One leg of the manometer is inclined at an angle θ, and the differential reading ℓ_2 is measured along the inclined tube. The difference in pressure $p_A - p_B$ can be expressed as

$$p_A + \gamma_1 h_1 - \gamma_2 \ell_2 \sin\theta - \gamma_3 h_3 = p_B$$

or

$$p_A - p_B = \gamma_2 \ell_2 \sin\theta + \gamma_3 h_3 - \gamma_1 h_1 \qquad \textbf{(2.15)}$$

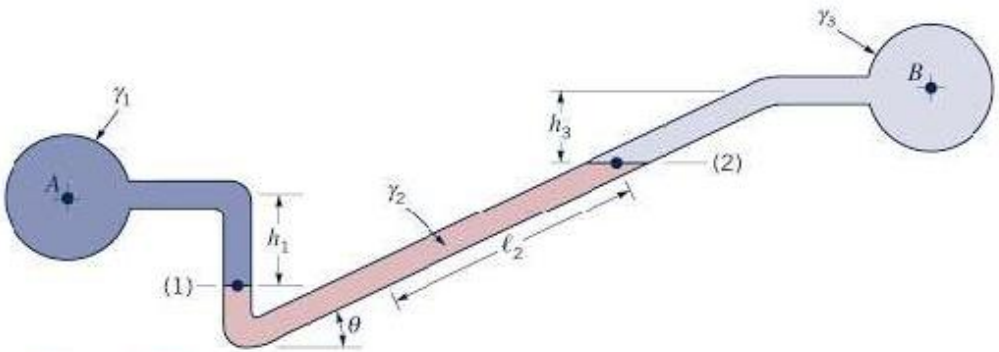


Figure 2.12 **Inclined-tube manometer.**

where it is to be noted the pressure difference between points (1) and (2) is due to the *vertical* distance between the points, which can be expressed as $\ell_2 \sin\theta$. Thus, for relatively small angles the differential reading along the inclined tube can be made large even for small pressure differences. The inclined-tube manometer is often used to measure small differences in gas pressures so that if pipes A and B contain a gas, then

$$p_A - p_B = \gamma_2 \ell_2 \sin\theta$$

or

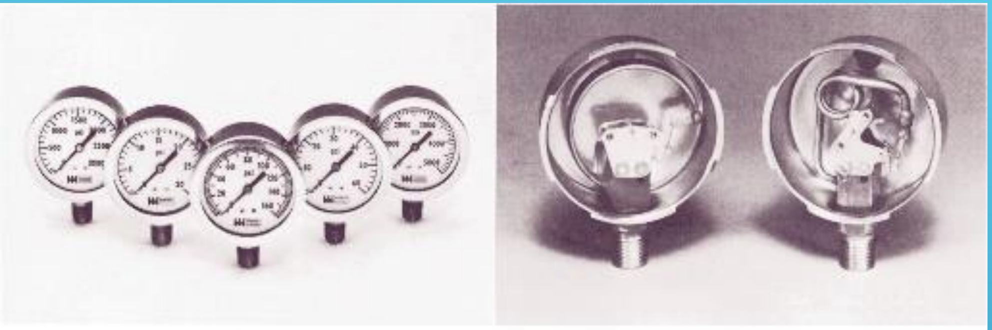
$$\ell_2 = \frac{p_A - p_B}{\gamma_2 \sin\theta} \quad \textbf{(2.16)}$$

where the contributions of the gas columns h_1 and h_3 have been neglected. Equation 2.16 and the figure in the margin show that the differential reading ℓ_2 (for a given pressure difference) of the inclined-tube manometer can be increased over that obtained with a conventional U-tube manometer by the factor $1/\sin\theta$. Recall that $\sin\theta \to 0$ as $\theta \to 0$.

2.7 Mechanical and Electronic Pressure-Measuring Devices

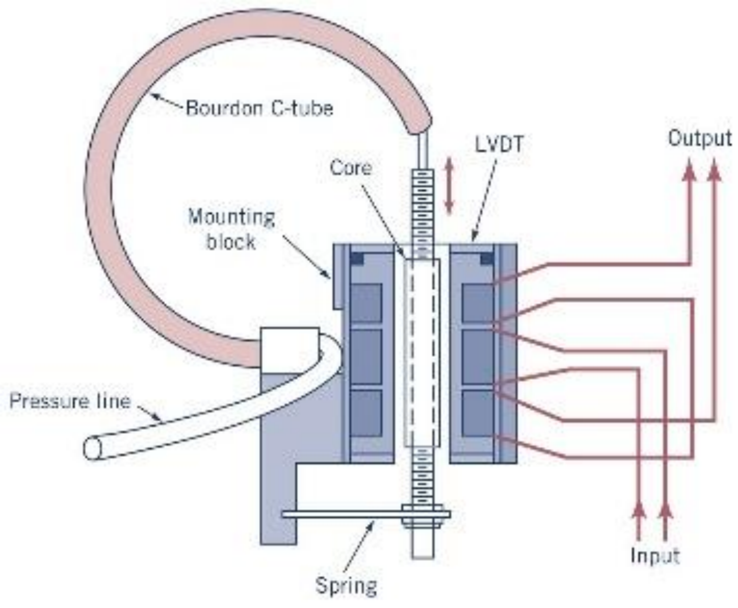
Manometers are widely used because they are simple, inexpensive, and reliable. However, they are limited in range, respond relatively slowly, are unsuitable for environments that might result in loss of gauge fluid, and are not easily interfaced with automated data acquisition systems. To overcome some of these problems numerous other types of pressure-measuring instruments have been developed. Most of these make use of the idea that when a pressure acts on an elastic structure the structure will deform, and this deformation can be related to the magnitude of the pressure. Probably the most familiar device of this kind is the ***Bourdon* pressure gage.**

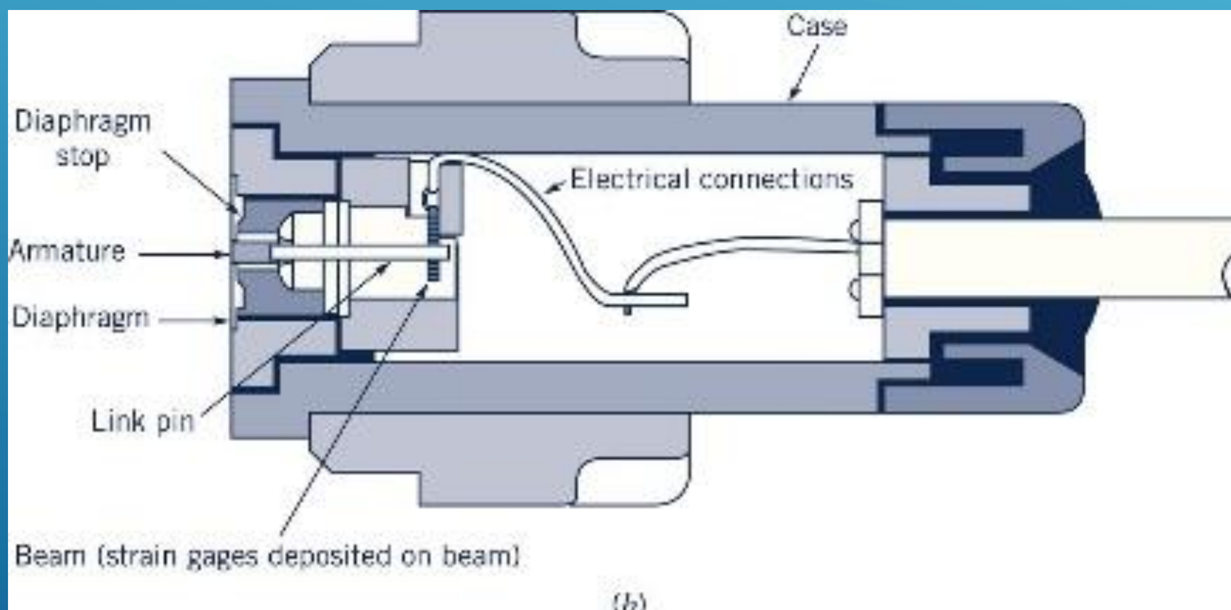
Mechanical and Electronic Pressure-Measuring Devices



(a) (b)

Photographs courtesy of Weiss Instruments, Inc.




Case
Diaphragm stop
Electrical connections
Armature
Diaphragm
Link pin
Beam (strain gages deposited on beam)
(b)

2.8 Hydrostatic Force on a Plane Surface

When a surface is submerged in a fluid, forces develop on the surface due to the fluid. The determination of these forces is important in the design of storage tanks, ships, dams, and other hydraulic structures. For fluids at rest we know that the force must be *perpendicular* to the surface since there are no shearing stresses present. We also know that the pressure will vary linearly with depth as shown in Fig. 2.16 if the fluid is incompressible. For a horizontal surface, such as the bottom of a liquid-filled tank (Fig. 2.16*a*), the magnitude of the resultant force is simply $F_R = pA$, where p is the uniform pressure on the bottom and A is the area of the bottom. For the open tank shown, $p = \gamma h$. Note that if atmospheric pressure acts on both sides of the bottom, as is illustrated, the *resultant* force on the bottom is simply due to the liquid in the tank. Since the pressure is constant and uniformly distributed over the bottom, the resultant force acts through the centroid of the area as shown in Fig. 2.16*a*. As shown in Fig. 2.16*b*, the pressure on the ends of the tank is not uniformly distributed. Determination of the force due to hydrostatic pressure on the tank ends is more challenging because the pressure is not constant, the ends may not be rectangular plates, and in general they may not be vertical.

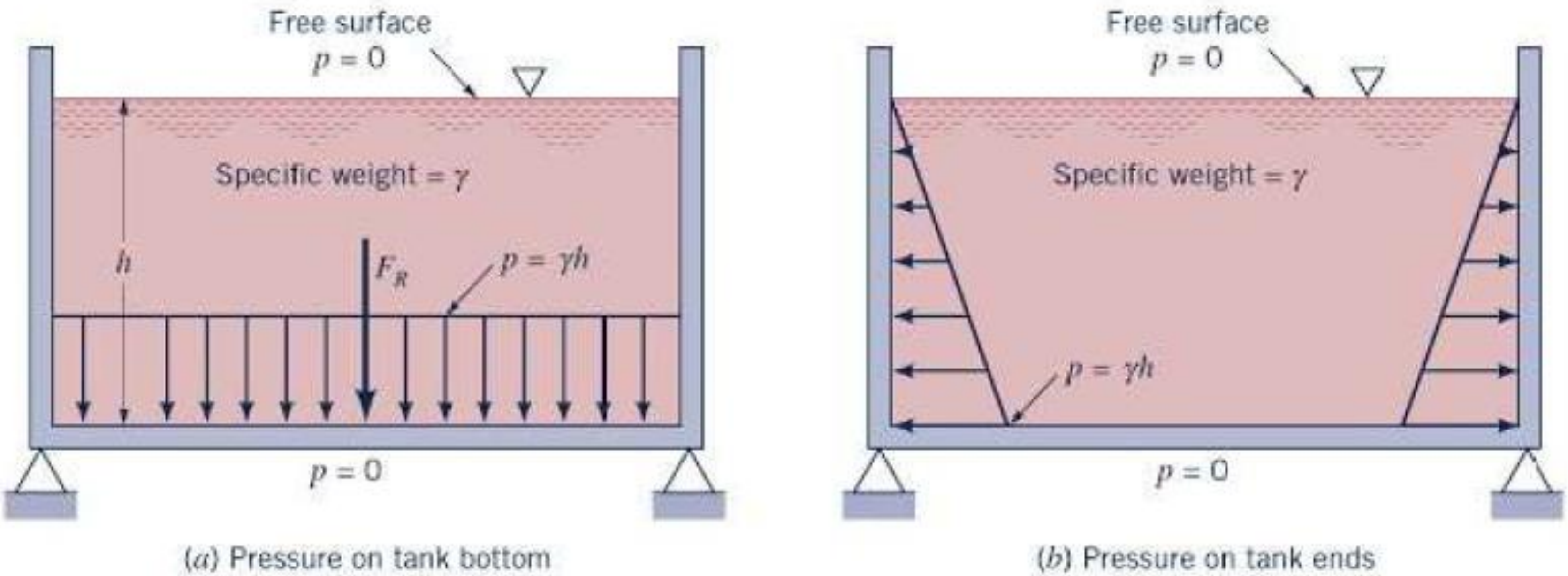


Figure 2.16 **(*a*) Pressure distribution and resultant hydrostatic force on the bottom of an open tank. (*b*) Pressure distribution on the ends of an open tank.**

$$F_R = \int_A \gamma h \, dA = \int_A \gamma y \sin\theta \, dA$$

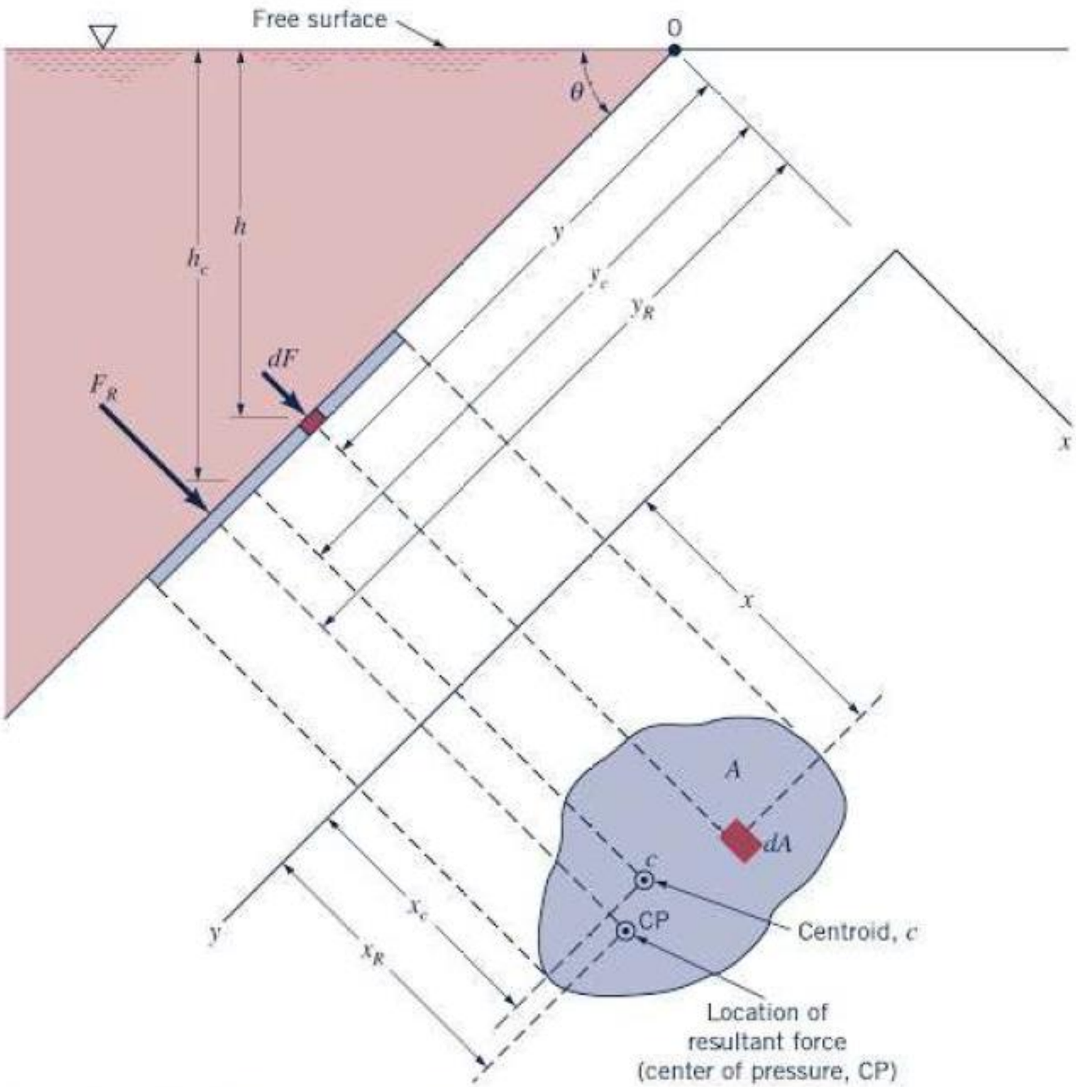


Figure 2.17 **Notation for hydrostatic force on an inclined plane surface of arbitrary shape.**

where $h = y \sin \theta$. For constant γ and θ

$$F_R = \gamma \sin \theta \int_A y \, dA \qquad \textbf{(2.17)}$$

The integral appearing in Eq. 2.17 is the *first moment of the area* with respect to the x axis, so we can write

$$\int_A y \, dA = y_c A$$

where y_c is the y coordinate of the centroid of area A measured from the x axis, which passes through 0. Equation 2.17 can thus be written as

$$F_R = \gamma A y_c \sin \theta$$

or more simply as

$$\boxed{F_R = \gamma h_c A} \qquad \textbf{(2.18)}$$

where, as shown by the figure in the margin, h_c is the vertical distance from the fluid surface to the centroid of the area. As indicated by the figure in the margin, it depends only on the specific weight of the fluid, the total area, and the depth of the centroid of the area below the surface. In effect, Eq. 2.18 indicates that the magnitude of the resultant force is equal to the pressure at the centroid of the area multiplied by the total area. Since all the differential forces that were summed to obtain F_R are perpendicular to the surface, the resultant F_R must also be perpendicular to the surface.

$$F_R y_R = \int_A y\, dF = \int_A \gamma \sin\theta\, y^2\, dA$$

and, noting that $F_R = \gamma A y_c \sin\theta$,

$$y_R = \frac{\int_A y^2\, dA}{y_c A}$$

The integral in the numerator is the *second moment of the area (moment of inertia),* I_x, with respect to an axis formed by the intersection of the plane containing the surface and the free surface (x axis). Thus, we can write

$$y_R = \frac{I_x}{y_c A}$$

Use can now be made of the parallel axis theorem to express I_x as

$$I_x = I_{xc} + A y_c^2$$

where I_{xc} is the second moment of the area with respect to an axis passing through its *centroid* and parallel to the x axis. Thus,

$$\boxed{y_R = \frac{I_{xc}}{y_c A} + y_c} \qquad \textbf{(2.19)}$$

The x coordinate, x_R, for the resultant force can be determined in a similar manner by summing moments about the y axis. Thus,

$$F_R x_R = \int_A \gamma \sin\theta \, xy \, dA$$

and, therefore,

$$x_R = \frac{\int_A xy \, dA}{y_c A} = \frac{I_{xy}}{y_c A}$$

where I_{xy} is the product of inertia with respect to the x and y axes. Again, using the parallel axis theorem,[1] we can write

$$x_R = \frac{I_{xyc}}{y_c A} + x_c \qquad (2.20)$$

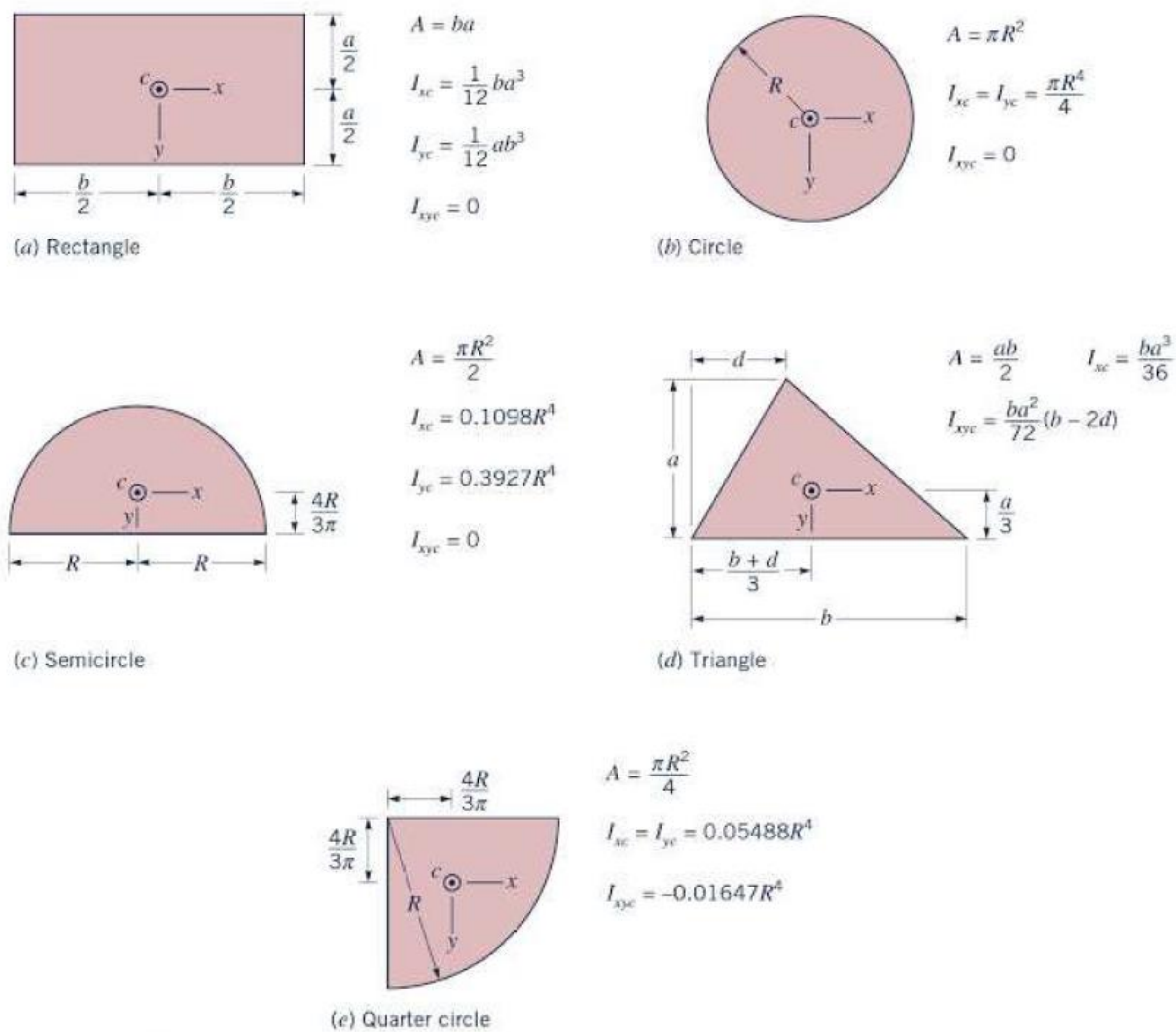


Figure 2.18 **Geometric properties of some common shapes.**

EXAMPLE 2.6 Hydrostatic Force on a Plane Circular Surface

GIVEN The 4-m-diameter circular gate of Fig. E2.6*a* is located in the inclined wall of a large reservoir containing water ($\gamma = 9.80\ \text{kN/m}^3$). The gate is mounted on a shaft along its horizontal diameter, and the water depth is 10 m at the shaft.

FIND Determine

(a) the magnitude and location of the resultant force exerted on the gate by the water and

(b) the moment that would have to be applied to the shaft to open the gate.

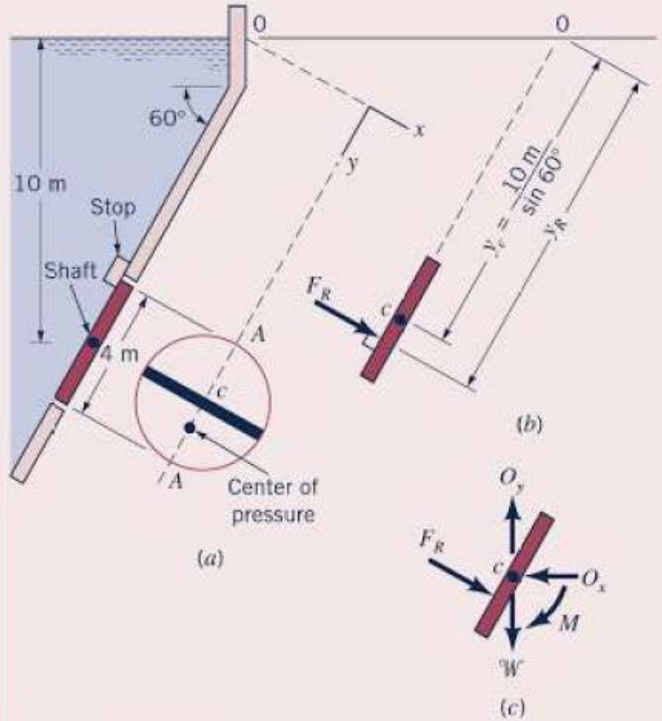


Figure E2.6*a–c*

SOLUTION

(a) To find the magnitude of the force of the water we can apply Eq. 2.18,

$$F_R = \gamma h_c A$$

and since the vertical distance from the fluid surface to the centroid of the area is 10 m, it follows that

$$F_R = (9.80 \times 10^3\ \text{N/m}^3)(10\ \text{m})(4\pi\ \text{m}^2)$$
$$= 1230 \times 10^3\ \text{N} = 1.23\ \text{MN} \qquad \textbf{(Ans)}$$

To locate the point (center of pressure) through which F_R acts, we use Eqs. 2.19 and 2.20,

$$x_R = \frac{I_{xyc}}{y_c A} + x_c \qquad y_R = \frac{I_{xc}}{y_c A} + y_c$$

For the coordinate system shown, $x_R = 0$ since the area is symmetric about the y-axis, and the center of pressure must lie along the diameter A-A. To obtain y_R, we have from Fig. 2.18

$$I_{xc} = \frac{\pi R^4}{4}$$

and y_c is shown in Fig. E2.6*b*. Thus,

$$y_R = \frac{(\pi/4)(2\text{ m})^4}{(10\text{ m}/\sin 60°)(4\pi\text{ m}^2)} + \frac{10\text{ m}}{\sin 60°}$$
$$= 0.0866\text{ m} + 11.55\text{ m} = 11.6\text{ m}$$

and the distance (along the gate) below the shaft to the center of pressure is

$$y_R - y_c = 0.0866\text{ m} \qquad \textbf{(Ans)}$$

We can conclude from this analysis that the force on the gate due to the water has a magnitude of 1.23 MN and acts through a point along its diameter A-A at a distance of 0.0866 m (along the gate) below the shaft. The force is perpendicular to the gate surface as shown in Fig. E2.6*b*.

COMMENT By repeating the calculations for various values of the depth to the centroid, h_c, the results shown in Fig. E2.6*d* are obtained. Note that as the depth increases, the distance between the center of pressure and the centroid decreases.

(b) The moment required to open the gate can be obtained with the aid of the free-body diagram of Fig. E2.6*c*. In this diagram $\mathcal{W}$ is the weight of the gate and O_x and O_y are the horizontal and vertical reactions of the shaft on the gate. We can now sum moments about the shaft

$$\sum M_c = 0$$

and, therefore,

$$M = F_R(y_R - y_c)$$
$$= (1230 \times 10^3\text{ N})(0.0866\text{ m})$$
$$= 1.07 \times 10^5\text{ N}\cdot\text{m} \qquad \textbf{(Ans)}$$

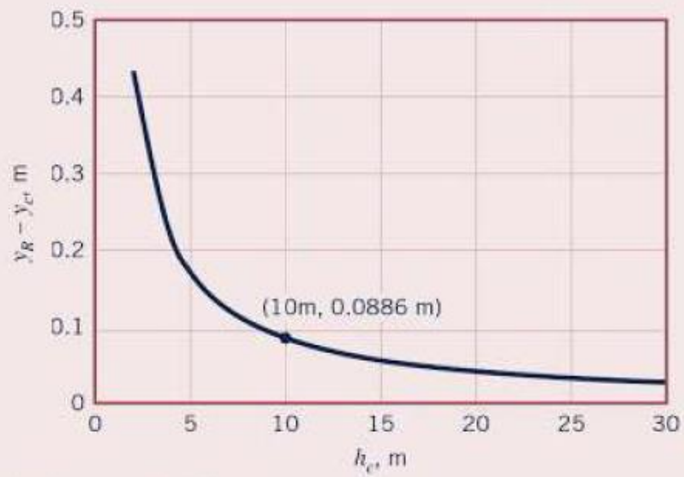


Figure E2.6*d*

2.9 Pressure Prism

An informative and useful graphical interpretation can be made for the force developed by a fluid acting on a plane rectangular area. Consider the pressure distribution along a vertical wall of a tank of constant width b, which contains a liquid having a specific weight γ. Since the pressure must vary linearly with depth, we can represent the variation as is shown in Fig. 2.19a, where the pressure is equal to zero at the upper surface and equal to γh at the bottom. It is apparent from this diagram that the average pressure occurs at the depth $h/2$ and, therefore, the resultant force acting on the rectangular area $A = bh$ is

$$F_R = p_{\text{av}}\, A = \gamma \left(\frac{h}{2}\right) A$$

$$F_R = \text{volume} = \frac{1}{2}(\gamma h)(bh) = \gamma \left(\frac{h}{2}\right) A$$

where bh is the area of the rectangular surface, A.

$$F_R = F_1 + F_2$$

where the components can readily be determined by inspection for rectangular surfaces. The location of F_R can be determined by summing moments about some convenient axis, such as one passing through A. In this instance

$$F_R y_A = F_1 y_1 + F_2 y_2$$

and y_1 and y_2 can be determined by inspection.

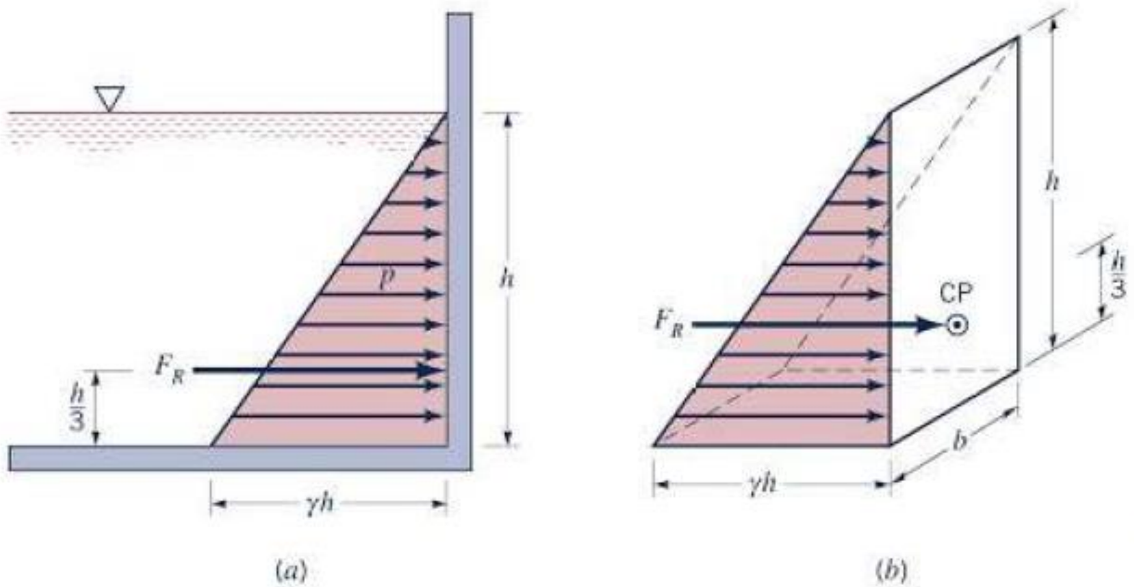


Figure 2.19 **Pressure prism for vertical rectangular area.**

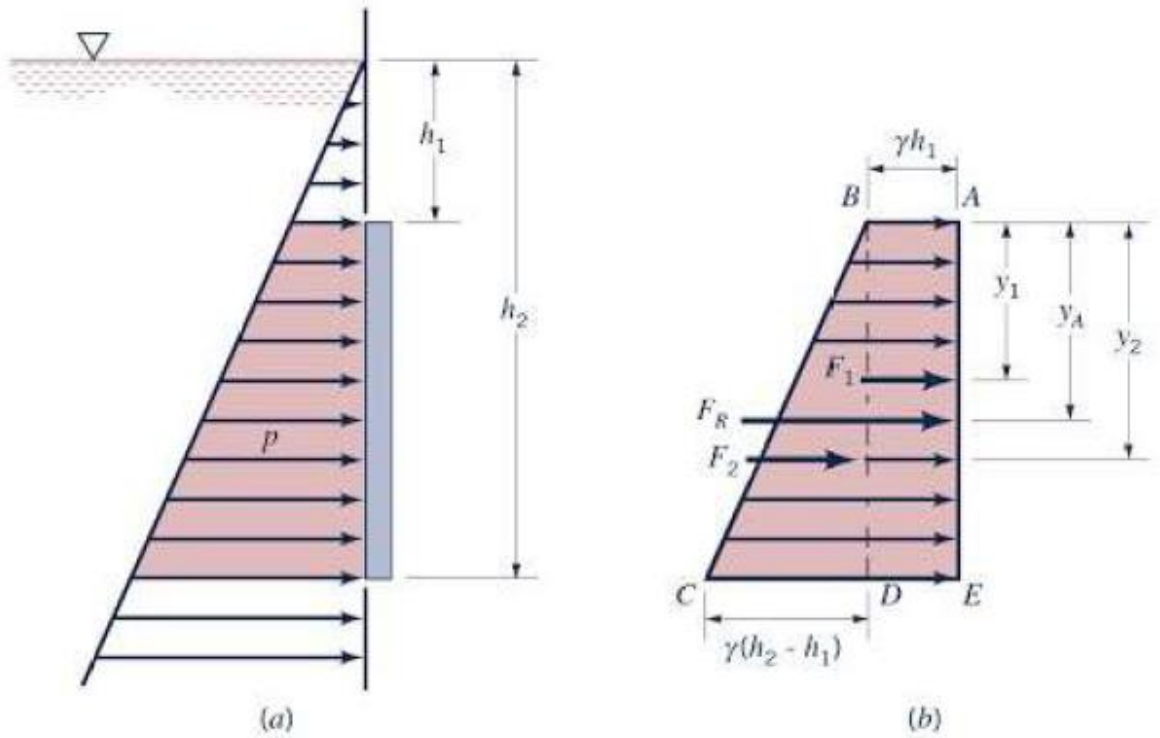


Figure 2.20 **Graphical representation of hydrostatic forces on a vertical rectangular surface.**

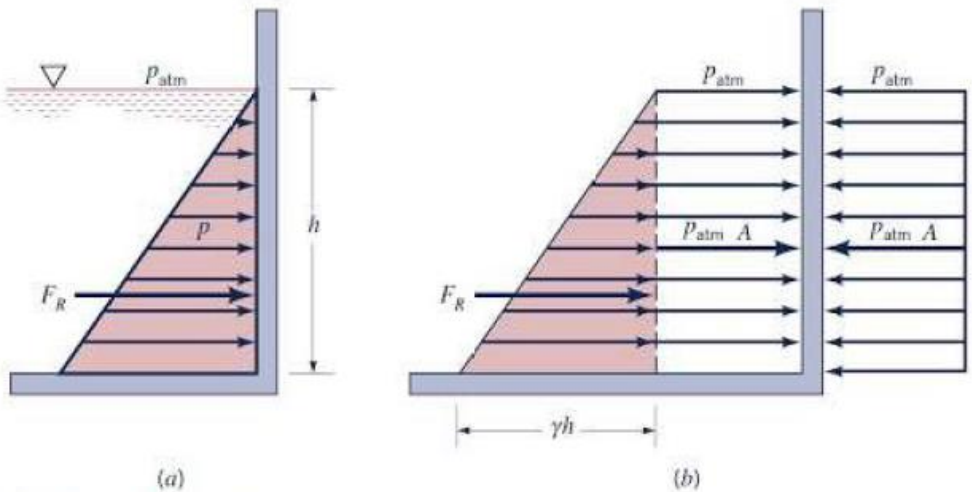


Figure 2.22 Effect of atmospheric pressure on the resultant force acting on a plane vertical wall.

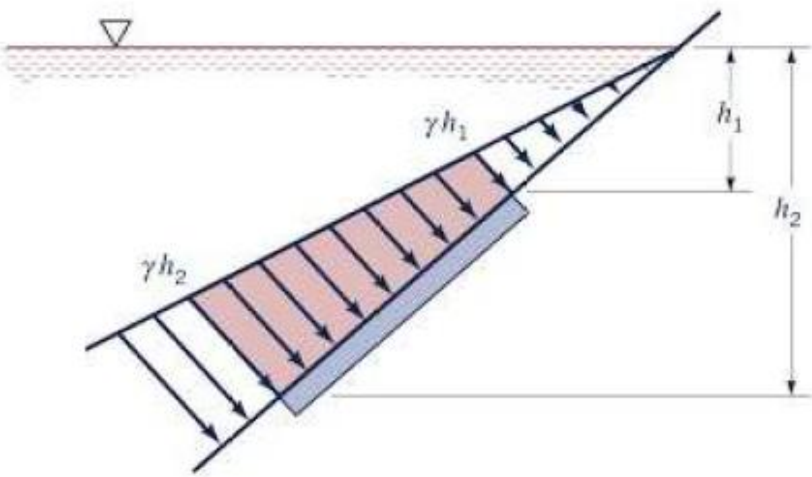


Figure 2.21 Pressure variation along an inclined plane area.

EXAMPLE 2.8 Use of the Pressure Prism Concept

GIVEN A pressurized tank contains oil ($SG = 0.90$) and has a square, 0.6-m by 0.6-m plate bolted to its side, as is illustrated in Fig. E2.8*a*. The pressure gage on the top of the tank reads 50 kPa, and the outside of the tank is at atmospheric pressure.

FIND What is the magnitude and location of the resultant force on the attached plate?

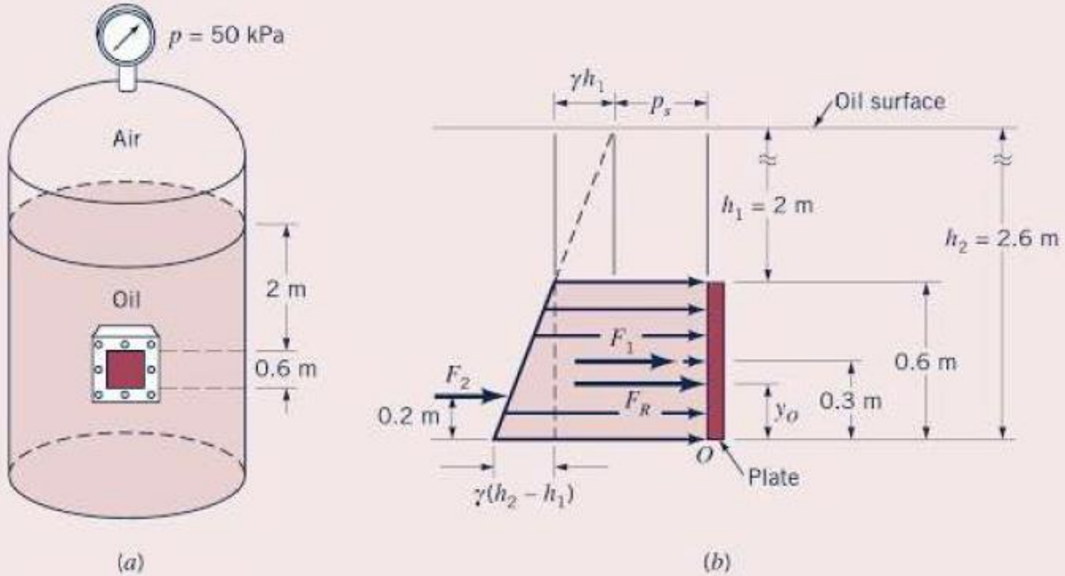


Figure E2.8

SOLUTION

The pressure distribution acting on the inside surface of the plate is shown in Fig. E2.8*b*. The pressure at a given point on the plate is due to the air pressure, p_s, at the oil surface and the pressure due to the oil, which varies linearly with depth as is shown in the figure. The resultant force on the plate (having an area A) is due to the components, F_1 and F_2, where F_1 and F_2 are due to the rectangular and triangular portions of the pressure distribution, respectively. Thus,

$$F_1 = (p_s + \gamma h_1)\, A$$
$$= [50 \times 10^3\ \text{N/m}^2 + (0.90)(9.81 \times 10^3\ \text{N/m}^3)(2\ \text{m})](0.36\ \text{m}^2)$$
$$= 24.4 \times 10^3\ \text{N}$$

and

$$F_2 = \gamma\left(\frac{h_2 - h_1}{2}\right)A$$
$$= (0.90)(9.81 \times 10^3\ \text{N/m}^3)\left(\frac{0.6\ \text{m}}{2}\right)(0.36\ \text{m}^2)$$
$$= 0.954 \times 10^3\ \text{N}$$

The magnitude of the resultant force, F_R, is therefore

$$F_R = F_1 + F_2 = 25.4 \times 10^3\ \text{N} = 25.4\ \text{kN} \qquad \textbf{(Ans)}$$

The vertical location of F_R can be obtained by summing moments around an axis through point O so that

$$F_R y_O = F_1(0.3\ \text{m}) + F_2(0.2\ \text{m})$$

or

$$y_O = \frac{(24.4 \times 10^3\ \text{N})(0.3\ \text{m}) + (0.954 \times 10^3\ \text{N})(0.2\ \text{m})}{25.4 \times 10^3\ \text{N}}$$
$$= 0.296\ \text{m} \qquad \textbf{(Ans)}$$

Thus, the force acts at a distance of 0.296 m above the bottom of the plate along the vertical axis of symmetry.

COMMENT Note that the air pressure used in the calculation of the force was gage pressure. Atmospheric pressure does not affect the resultant force (magnitude or location), since it acts on both sides of the plate, thereby canceling its effect.

2.10 Hydrostatic Force on a Curved Surface

The equations developed in Section 2.8 for the magnitude and location of the resultant force acting on a submerged surface only apply to plane surfaces. However, many surfaces of interest (such as those associated with dams, pipes, and tanks) are nonplanar. The domed bottom of the beverage bottle shown in the figure in the margin shows a typical curved surface example. Although the resultant fluid force can be determined by integration, as was done for the plane surfaces, this is generally a rather tedious process and no simple, general formulas can be developed. As an alternative approach, we will consider the equilibrium of the fluid volume enclosed by the curved surface of interest and the horizontal and vertical projections of this surface.

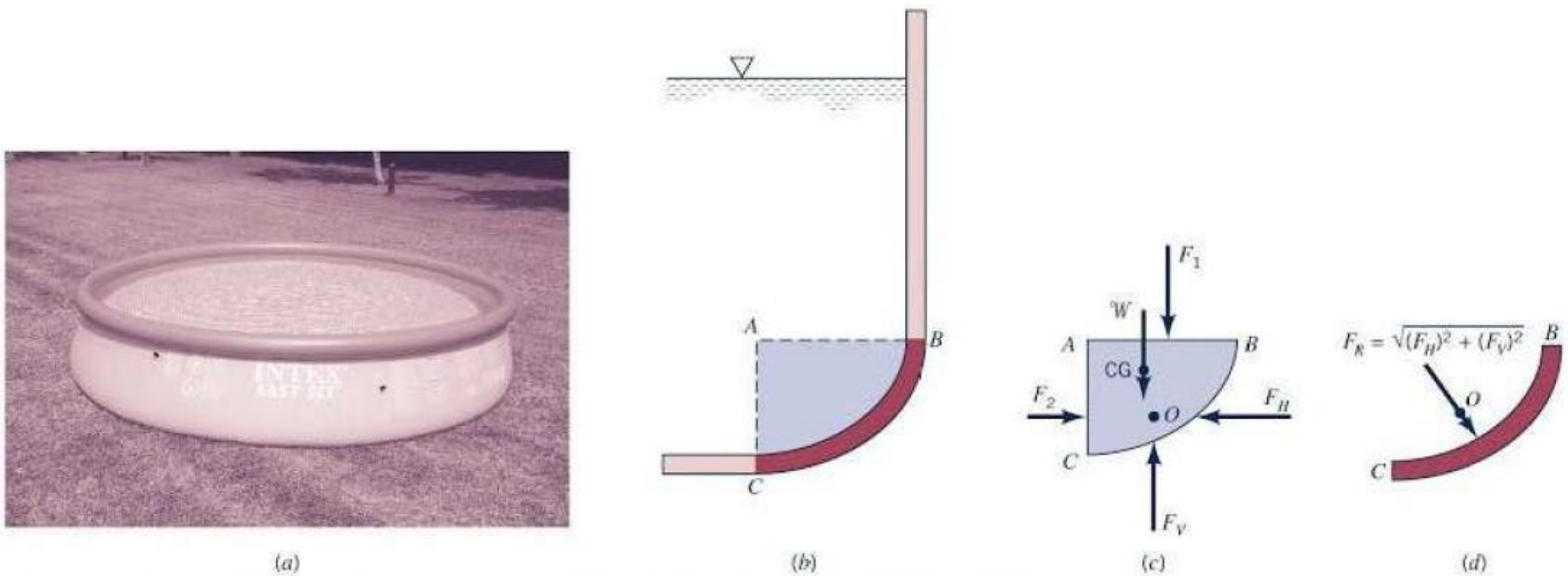


Figure 2.23 Hydrostatic force on a curved surface. (Photograph courtesy of Intex Marketing, Ltd.)

In order for this force system to be in equilibrium, the horizontal component F_H must be equal in magnitude and collinear with F_2, and the vertical component F_V equal in magnitude and collinear with the resultant of the vertical forces F_1 and $\mathscr{W}$. This follows since the three forces acting on the fluid mass (F_2, the resultant of F_1 and $\mathscr{W}$, and the resultant force that the tank exerts on the mass) must form a *concurrent* force system. That is, from the principles of statics, it is known that when a body is held in equilibrium by three nonparallel forces, they must be concurrent (their lines of action intersect at a common point) and coplanar. Thus,

$$F_H = F_2$$

$$F_V = F_1 + \mathscr{W}$$

and the magnitude of the resultant is obtained from the equation

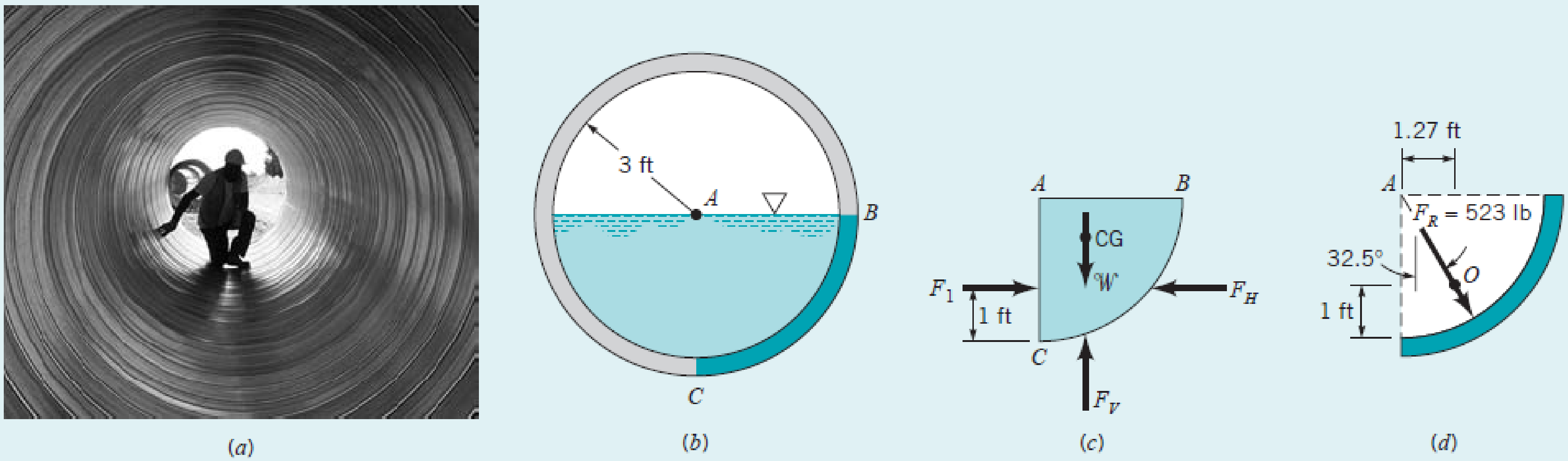
$$F_R = \sqrt{(F_H)^2 + (F_V)^2}$$

The resultant F_R passes through the point O, which can be located by summing moments about an appropriate axis. The resultant force of the fluid acting *on the curved surface BC* is equal and opposite in direction to that obtained from the free-body diagram of Fig. 2.23*c*. The desired fluid force is shown in Fig. 2.23*d*.

EXAMPLE 2.9 Hydrostatic Pressure Force on a Curved Surface

GIVEN A 6-ft-diameter drainage conduit of the type shown in Fig. E2.9*a* is half full of water at rest, as shown in Fig. E2.9*b*.

FIND Determine the magnitude and line of action of the resultant force that the water exerts on a 1-ft length of the curved section *BC* of the conduit wall.



■ FIGURE E2.9 (Photograph courtesy of CONTECH Construction Products, Inc.)

Solution

We first isolate a volume of fluid bounded by the curved section *BC*, the horizontal surface *AB*, and the vertical surface *AC*, as shown in Fig. E2.9*c*. The volume has a length of 1 ft. The forces acting on the volume are the horizontal force, F_1, which acts on the vertical surface *AC*, the weight, $\mathcal{W}$, of the fluid contained within the volume, and the horizontal and vertical components of the force of the conduit wall on the fluid, F_H and F_V, respectively.

The magnitude of F_1 is found from the equation

$$F_1 = \gamma h_c A = (62.4 \text{ lb/ft}^3)(\tfrac{3}{2} \text{ ft})(3 \text{ ft}^2) = 281 \text{ lb}$$

and this force acts 1 ft above *C* as shown. The weight $\mathcal{W} = \gamma \mathcal{V}$, where $\mathcal{V}$ is the fluid volume, is

$$\mathcal{W} = \gamma \mathcal{V} = (62.4 \text{ lb/ft}^3)(9\pi/4 \text{ ft}^2)(1 \text{ ft}) = 441 \text{ lb}$$

and acts through the center of gravity of the mass of fluid, which according to Fig. 2.18 is located 1.27 ft to the right of *AC* as shown. Therefore, to satisfy equilibrium

$$F_H = F_1 = 281 \text{ lb} \qquad F_V = \mathcal{W} = 441 \text{ lb}$$

and the magnitude of the resultant force is

$$F_R = \sqrt{(F_H)^2 + (F_V)^2}$$
$$= \sqrt{(281 \text{ lb})^2 + (441 \text{ lb})^2} = 523 \text{ lb} \qquad \textbf{(Ans)}$$

The force the water exerts *on* the conduit wall is equal, but *opposite in direction,* to the forces F_H and F_V shown in Fig. E2.9*c*. Thus, the resultant force *on the conduit wall* is shown in Fig. E2.9*d*. This force acts through the point *O* at the angle shown.

2.35 The rigid gate, *OAB*, of Fig. P2.35 is hinged at *O* and rests against a rigid support at *B*. What minimum horizontal force, *P*, is required to hold the gate closed if its width is 3 m? Neglect the weight of the gate and friction in the hinge. The back of the gate is exposed to the atmosphere.

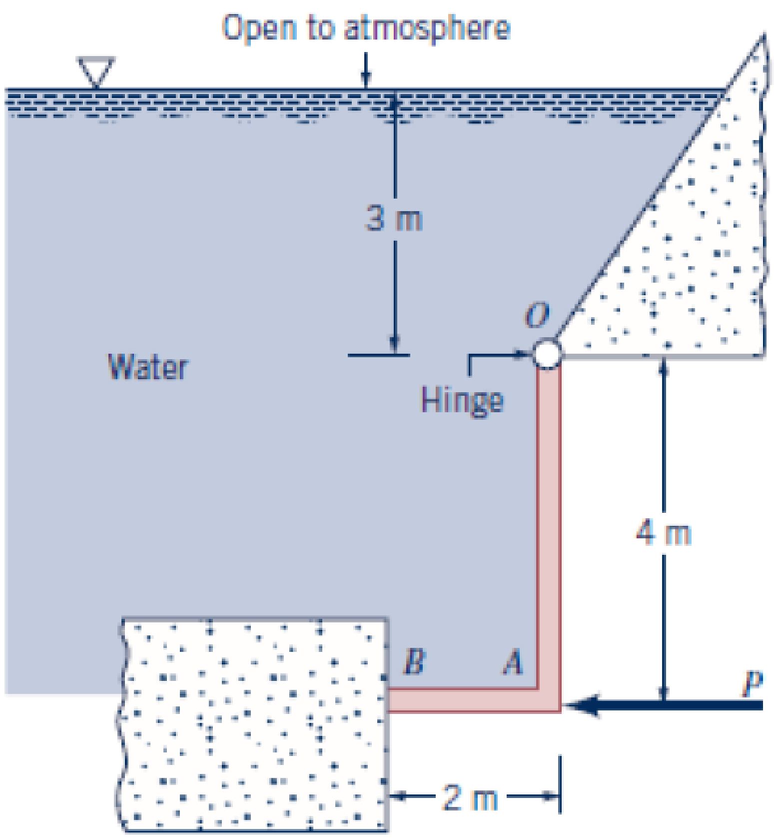


Figure P2.35

$$F_1 = \gamma h_{c_1} A_1 \quad \text{where } h_{c_1} = 5\,m$$

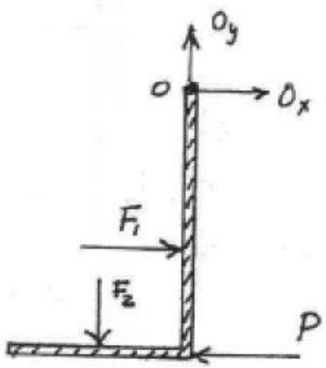
Thus,

$$F_1 = \left(9800 \frac{N}{m^3}\right)(5\,m)(4\,m \times 3\,m)$$
$$= 5.88 \times 10^5\,N$$

$$F_2 = \gamma h_{c_2} A_2 \quad \text{where } h_{c_2} = 7\,m$$

so that

$$F_2 = \left(9800 \frac{N}{m^3}\right)(7\,m)(2\,m \times 3\,m)$$
$$= 4.12 \times 10^5\,N$$



To locate F_1,

$$y_{R_1} = \frac{I_{xc}}{y_{c_1} A_1} + y_{c_1} = \frac{\frac{1}{12}(3\,m)(4\,m)^3}{(5\,m)(4\,m \times 3\,m)} + 5\,m = 5.267\,m$$

The force F_2 acts at the center of the AB section. Thus,

$$\Sigma M_O = 0$$

and

$$F_1(5.267\,m - 3\,m) + F_2(1\,m) = P(4\,m)$$

so that

$$P = \frac{(5.88 \times 10^5\,N)(2.267\,m) + (4.12 \times 10^5\,N)(1\,m)}{4\,m}$$

$$= \underline{\underline{436\,kN}}$$

2.45 A tank wall has the shape shown in Fig. P2.45. Determine the horizontal and vertical components of the force of the water on a 4-ft length of the curved section *AB*.

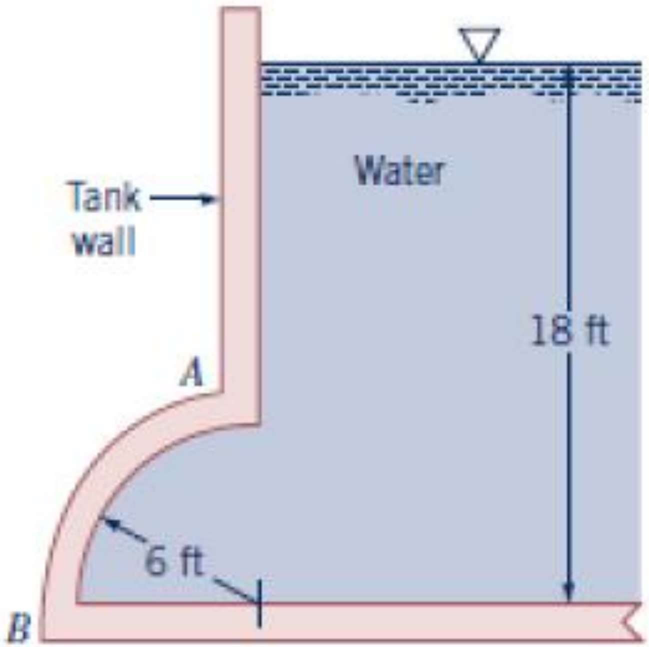


Figure P2.45

$F_1 = \gamma h_{c1} A_1$

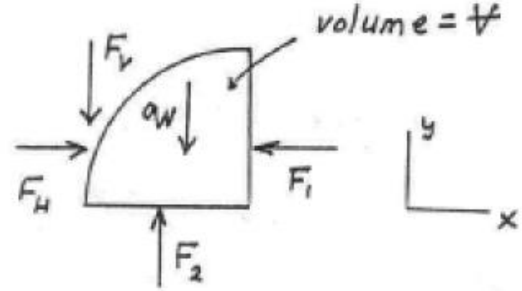
$= (62.4 \frac{lb}{ft^3})(15 ft)(6 ft \times 4 ft)$

$= 22,500 \ lb$

$F_2 = \gamma h_{c2} A_2$

$= (62.4 \frac{lb}{ft^3})(18 ft)(6 ft \times 4 ft)$

$= 27,000 \ lb$



$\mathcal{W} = \gamma \forall = (62.4 \frac{lb}{ft^3})(\frac{1}{4})(\pi)(6 ft)^2(4 ft)$

$= 7060 \ lb$

For equilibrium,

$\Sigma F_x = 0$

so that

$F_H = F_1 = \underline{\underline{22,500 \ lb \leftarrow \text{ on tank}}}$

and

$F_V = F_2 - \mathcal{W} = 27,000 \ lb - 7060 \ lb = \underline{\underline{19,900 \ lb \uparrow \text{ on tank}}}$

2.11 Buoyancy, Flotation, and Stability

2.11.1 Archimedes' Principle

When a stationary body is completely submerged in a fluid (such as the hot air balloon shown in the figure in the margin), or floating so that it is only partially submerged, the resultant fluid force acting on the body is called the ***buoyant force.*** A net upward vertical force results because pressure increases with depth and the pressure forces acting from below are larger than the pressure forces acting from above. This force can be determined through an approach similar to that used in the previous section for forces on curved surfaces. Consider a body of arbitrary shape, having a volume $\forall$, that is immersed in a fluid as illustrated in Fig. 2.24*a*. We enclose the body in a parallelepiped and draw a free-body diagram of the parallelepiped with the body removed as shown in Fig. 2.24*b*. Note that the forces F_1, F_2, F_3, and F_4 are simply the forces exerted on the plane surfaces of the parallelepiped (for simplicity the forces in the x direction are not shown), $\mathcal{W}$ is the weight of the shaded fluid volume (parallelepiped minus body), and F_B is the force the body is exerting *on the fluid*. The forces on the vertical surfaces, such as F_3 and F_4, are all equal and cancel, so the equilibrium equation of interest is in the z direction and can be expressed as

$$F_B = F_2 - F_1 - \mathcal{W} \tag{2.21}$$

If the specific weight of the fluid is constant, then

$$F_2 - F_1 = \gamma(h_2 - h_1)A$$

where A is the horizontal area of the upper (or lower) surface of the parallelepiped, and Eq. 2.21 can be written as

$$F_B = \gamma(h_2 - h_1)A - \gamma[(h_2 - h_1)A - \forall]$$

Simplifying, we arrive at the desired expression for the buoyant force

$$\boxed{F_B = \gamma \forall} \tag{2.22}$$

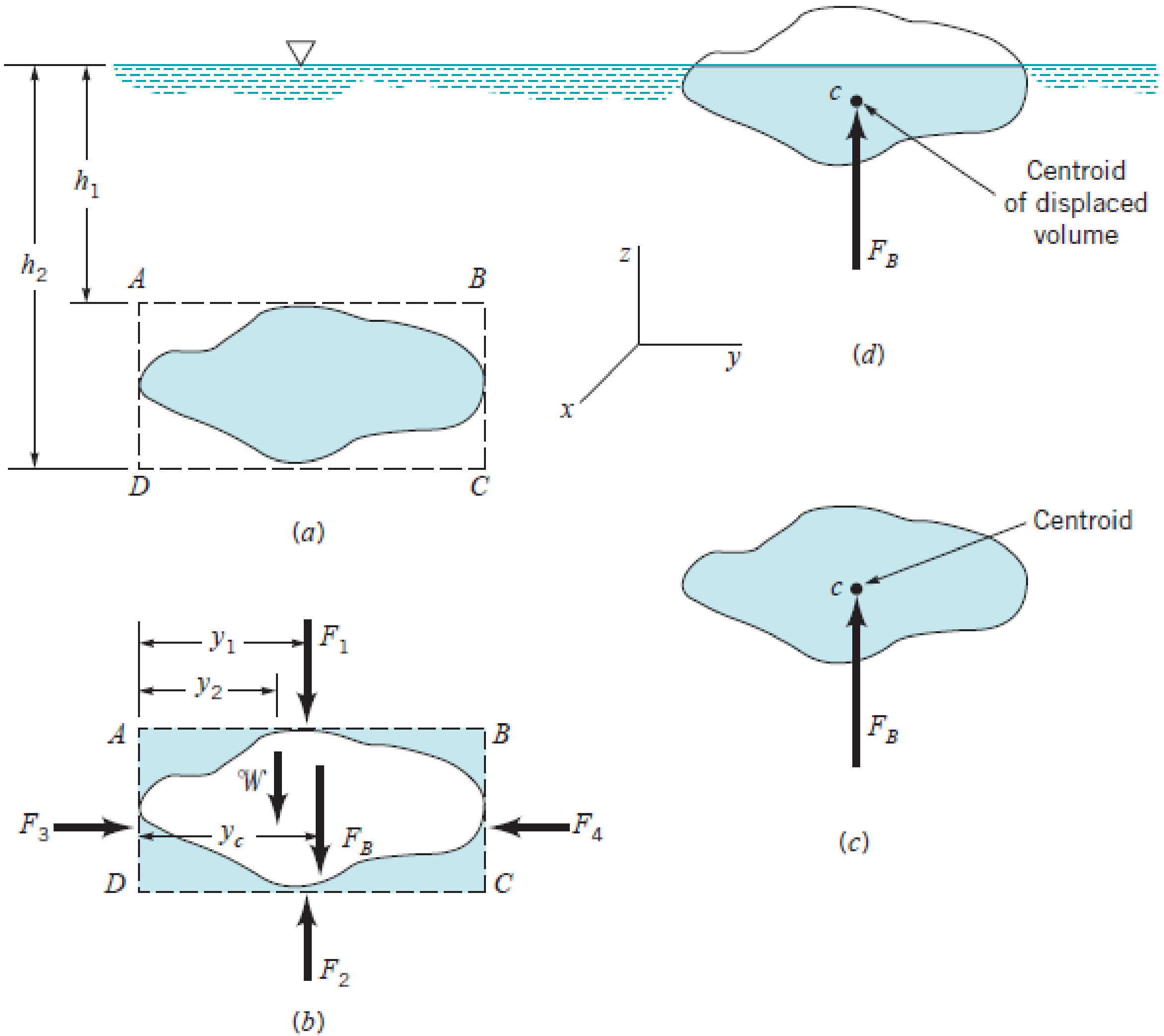


FIGURE 2.24 Buoyant force on submerged and floating bodies.

The location of the line of action of the buoyant force can be determined by summing moments of the forces shown on the free-body diagram in Fig. 2.24*b* with respect to some convenient axis. For example, summing moments about an axis perpendicular to the paper through point *D* we have

$$F_B y_c = F_2 y_1 - F_1 y_1 - \mathcal{W} y_2$$

and on substitution for the various forces

$$\forall y_c = \forall_T y_1 - (\forall_T - \forall) y_2 \qquad \textbf{(2.23)}$$

where $\forall_T$ is the total volume $(h_2 - h_1)A$. The right-hand side of Eq. 2.23 is the first moment of the displaced volume $\forall$ with respect to the x–z plane so that y_c is equal to the y coordinate of the centroid of the volume $\forall$. In a similar fashion it can be shown that the x coordinate of the buoyant force coincides with the x coordinate of the centroid. Thus, we conclude that the *buoyant force passes through the centroid of the displaced volume* as shown in Fig. 2.24*c*. The point through which the buoyant force acts is called the ***center of buoyancy.***

EXAMPLE 2.10 Buoyant Force on a Submerged Object

GIVEN A Type I offshore life jacket (personal flotation device) of the type worn by commercial fishermen is shown in Fig. E2.10*a*. It is designed for extended survival in rough, open water. According to U.S. Coast Guard regulations, the life jacket must provide a minimum 22-lb net upward force on the user. Consider such a life jacket that uses a foam material with a specific weight of 2.0 lb/ft^3 for the main flotation material. The remaining material (cloth, straps, fasteners, etc.) weighs 1.3 lb and is of negligible volume.

FIND Determine the minimum volume of foam needed for this life jacket.

Figure E2.10*a*

SOLUTION

A free-body diagram of the life jacket is shown in Fig. E2.10*b*, where F_B is the buoyant force acting on the life jacket, $\mathcal{W}_F$ is the weight of the foam, $\mathcal{W}_S = 1.3$ lb is the weight of the remaining material, and $F_U = 22$ lb is the required force on the user. For equilibrium it follows that

$$F_B = \mathcal{W}_F + \mathcal{W}_S + F_U \qquad (1)$$

where from Eq. 2.22

$$F_B = \gamma_{\text{water}} \, \mathcal{V}$$

Here $\gamma_{\text{water}} = 64.0$ lb/ft^3 is the specific weight of seawater and $\mathcal{V}$ is the volume of the foam. Also $\mathcal{W}_{\text{foam}} = \gamma_{\text{foam}} \, \mathcal{V}$, where $\gamma_{\text{foam}} = 2.0$ lb/ft^3 is the specific weight of the foam. Thus, from Eq. 1

or

$$\gamma_{water} V = \gamma_{foam} V + W_S + F_U$$

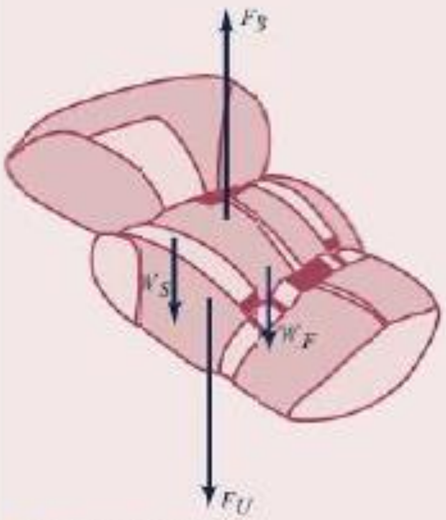


Figure E2.10*b*

$$V = (W_S + F_U)/(\gamma_{water} - \gamma_{foam})$$
$$= (1.3 \text{ lb} + 22 \text{ lb})/(64.0 \text{ lb/ft}^3 - 2.0 \text{ lb/ft}^3)$$
$$= 0.376 \text{ ft}^3 \quad \textbf{(Ans)}$$

COMMENTS In this example, rather than using difficult-to-calculate hydrostatic pressure force on the irregularly shaped life jacket, we have used the buoyant force. The net effect of the pressure forces on the surface of the life jacket is equal to the upward buoyant force. Do not include both the buoyant force and the hydrostatic pressure effects in your calculations—use one or the other.

There is more to the proper design of a life jacket than just the volume needed for the required buoyancy. According to regulations, a Type I life jacket must also be designed so that it provides proper protection to the user by turning an unconscious person in the water to a face-up position as shown in Fig. E2.10*a*. This involves the concept of the stability of a floating object (see Section 2.11.2). The life jacket should also provide minimum interference under ordinary working conditions so as to encourage its use by commercial fishermen.

2.50 A 1-m-diameter cylindrical mass, M, is connected to a 2-m-wide rectangular gate as shown in Fig.2.50. The gate is to open when the water level, h, drops below 2.5 m. Determine the required value for M. Neglect friction at the gate hinge and the pulley.

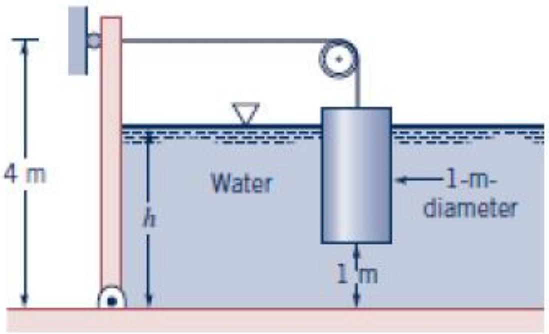


Figure P2.50

$$F_R = \gamma h_c A$$

$$= \gamma \left(\frac{h}{2}\right) h\,(2)$$

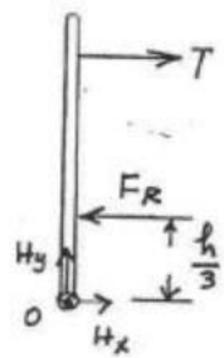
$$= \gamma h^2$$

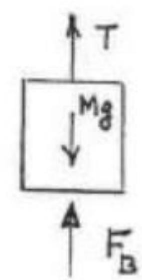
where all lengths are in m.

For equilibrium,

$$\sum M_o = 0$$

so that





$$4T = \left(\frac{h}{3}\right) F_R = \gamma \frac{h^3}{3}$$

and

$$T = \frac{\gamma h^3}{12}$$

For the cylindrical mass $\Sigma F_{vertical} = 0$ and

$$T = Mg - F_B = Mg - \gamma \forall_{mass}$$

Thus,

$$M = \frac{T + \gamma \forall_{mass}}{g} = \frac{\frac{\gamma h^3}{12} + \gamma \left(\frac{\pi}{4}\right)(1)^2 (h-1)}{g}$$

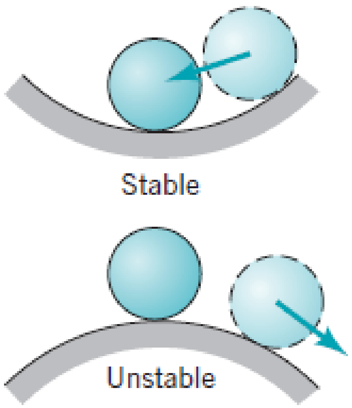
and for $h = 2.5m$

$$M = \frac{\left(9.80 \times 10^3 \frac{N}{m^3}\right)\left[\frac{(2.5m)^3}{12} + \frac{\pi}{4}(1m)^2(2.5m - 1.0m)\right]}{9.81 \frac{m}{s^2}}$$

$$= \underline{\underline{2480 \text{ kg}}}$$

2.11.2 Stability

Another interesting and important problem associated with submerged or floating bodies is concerned with the stability of the bodies. As illustrated by the figure in the margin, a body is said to be in a *stable equilibrium* position if, when displaced, it returns to its equilibrium position. Conversely, it is in an *unstable equilibrium* position if, when displaced (even slightly), it moves to a new equilibrium position. Stability considerations are particularly important for submerged or floating bodies since the centers of buoyancy and gravity do not necessarily coincide. A small rotation can result in either a restoring or overturning couple. For example, for the *completely* submerged body shown in Fig. 2.25, which has a center of gravity below the center of buoyancy, a rotation from its equilibrium position will create a restoring couple formed by the weight, $\mathcal{W}$, and the buoyant force, F_B, which causes the body to rotate back to its original position. Thus, for this configuration the body is stable. It is to be noted that as long as the center of gravity falls *below* the center of buoyancy, this will always be true; that is, the body is in a *stable equilibrium* position with respect to small rotations. However, as is illustrated in Fig. 2.26, if the center of gravity of the completely submerged body is above the center of buoyancy, the resulting couple formed by the weight and the buoyant force will cause the body to overturn and move to a new equilibrium position. Thus, a completely submerged body with its center of gravity *above* its center of buoyancy is in an *unstable equilibrium* position.



The stability of a body can be determined by considering what happens when it is displaced from its equilibrium position.

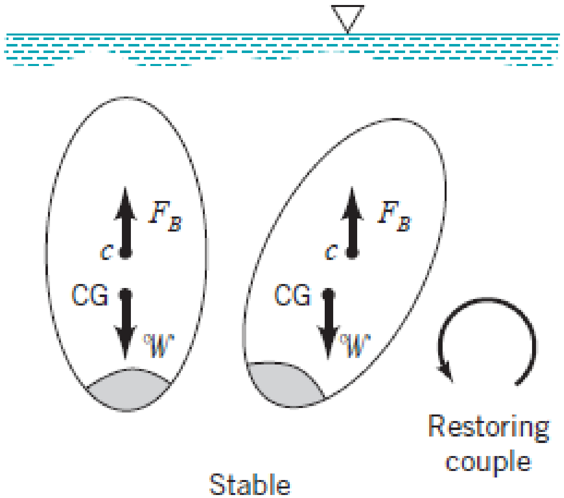


FIGURE 2.25
Stability of a completely immersed body—center of gravity below centroid.

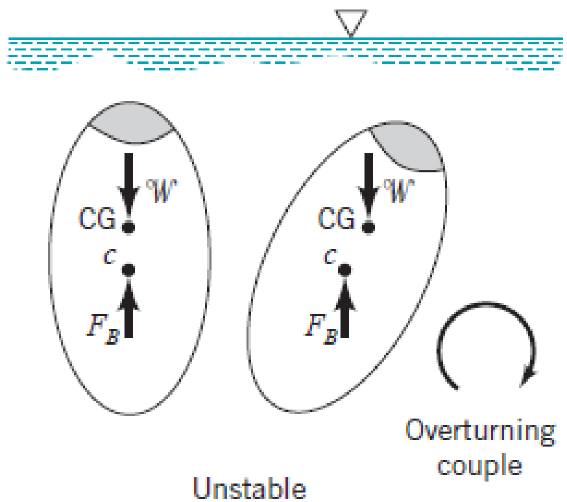


FIGURE 2.26
Stability of a completely immersed body—center of gravity above centroid.

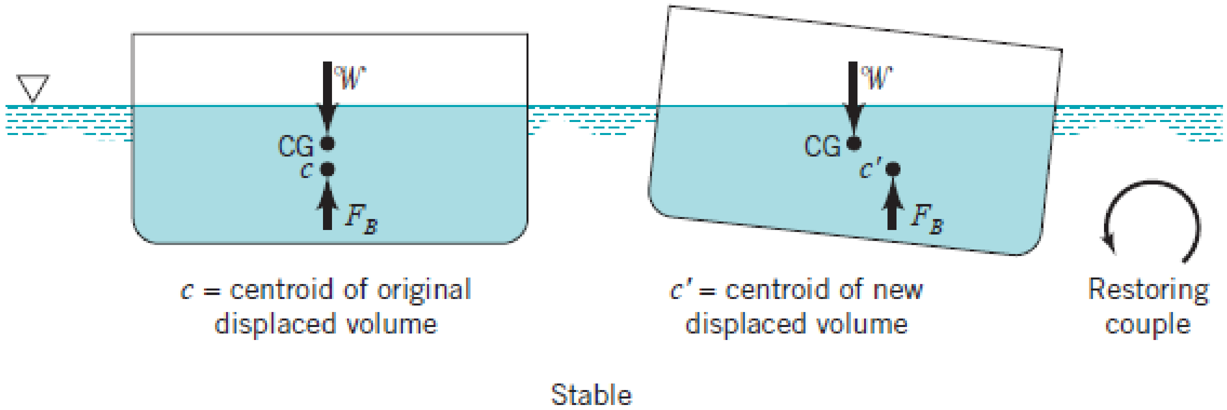


FIGURE 2.27 Stability of a floating body—stable configuration.

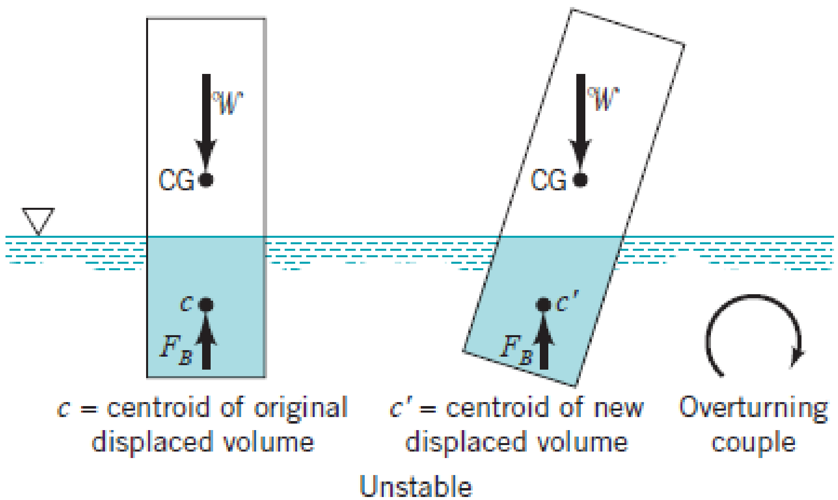


FIGURE 2.28 Stability of a floating body—unstable configuration.

2.12 Pressure Variation in a Fluid with Rigid-Body Motion

Although in this chapter we have been primarily concerned with fluids at rest, the general equation of motion (Eq. 2.2)

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \mathbf{a}$$

was developed for both fluids at rest and fluids in motion, with the only stipulation being that there were no shearing stresses present. Equation 2.2 in component form, based on rectangular coordinates with the positive z axis being vertically upward, can be expressed as

$$-\frac{\partial p}{\partial x} = \rho a_x \qquad -\frac{\partial p}{\partial y} = \rho a_y \qquad -\frac{\partial p}{\partial z} = \gamma + \rho a_z \qquad \textbf{(2.24)}$$

A general class of problems involving fluid motion in which there are no shearing stresses occurs when a mass of fluid undergoes rigid-body motion. For example, if a container of fluid accelerates along a straight path, the fluid will move as a rigid mass (after the initial sloshing motion has died out) with each particle having the same acceleration. Since there is no deformation,

2.12.1 Linear Motion

We first consider an open container of a liquid that is translating along a straight path with a constant acceleration **a** as illustrated in Fig. 2.29. Since $a_x = 0$, it follows from the first of Eqs. 2.24 that the pressure gradient in the x direction is zero $(\partial p/\partial x = 0)$. In the y and z directions

$$\frac{\partial p}{\partial y} = -\rho a_y \tag{2.25}$$

$$\frac{\partial p}{\partial z} = -\rho(g + a_z) \tag{2.26}$$

The change in pressure between two closely spaced points located at y, z, and $y + dy$, $z + dz$ can be expressed as

$$dp = \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz$$

or in terms of the results from Eqs. 2.25 and 2.26

$$dp = -\rho a_y\,dy - \rho(g + a_z)\,dz \tag{2.27}$$

Along a line of *constant* pressure, $dp = 0$, and therefore from Eq. 2.27 it follows that the slope of this line is given by the relationship

$$\frac{dz}{dy} = -\frac{a_y}{g + a_z} \tag{2.28}$$

This relationship is illustrated by the figure in the margin. Along a free surface the pressure is constant, so that for the accelerating mass shown in Fig. 2.29 the free surface will be inclined if $a_y \neq 0$. In addition, all lines of constant pressure will be parallel to the free surface as illustrated.

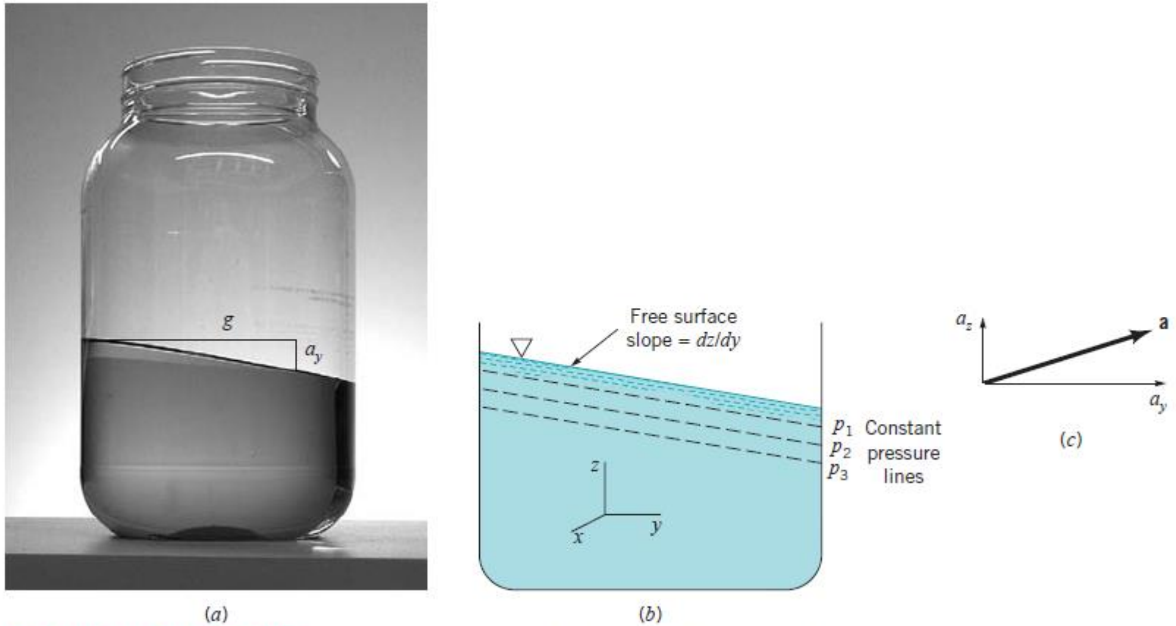


FIGURE 2.29 Linear acceleration of a liquid with a free surface.

For the special circumstance in which $a_y = 0$, $a_z \neq 0$, which corresponds to the mass of fluid accelerating in the vertical direction, Eq. 2.28 indicates that the fluid surface will be horizontal. However, from Eq. 2.26 we see that the pressure distribution is not hydrostatic, but is given by the equation

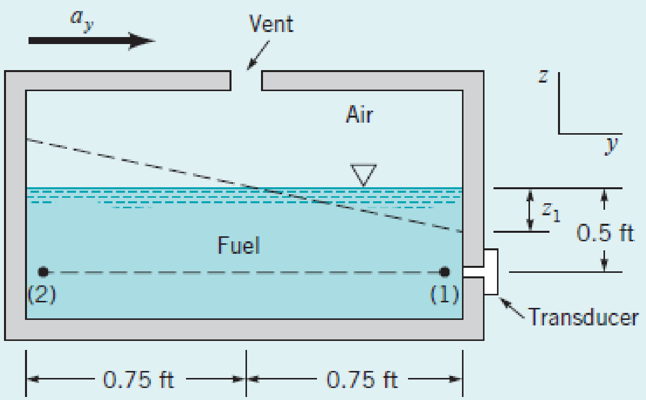
$$\frac{dp}{dz} = -\rho(g + a_z)$$

For fluids of constant density this equation shows that the pressure will vary linearly with depth, but the variation is due to the combined effects of gravity and the externally induced acceleration, $\rho(g + a_z)$, rather than simply the specific weight ρg. Thus, for example, the pressure along the bottom of a liquid-filled tank which is resting on the floor of an elevator that is accelerating upward will be increased over that which exists when the tank is at rest (or moving with a constant velocity). It is to be noted that for a *freely falling* fluid mass ($a_z = -g$), the pressure gradients in all three coordinate directions are zero, which means that if the pressure surrounding the mass is zero,

EXAMPLE 2.11 Pressure Variation in an Accelerating Tank

GIVEN The cross section for the fuel tank of an experimental vehicle is shown in Fig. E2.11. The rectangular tank is vented to the atmosphere and the specific gravity of the fuel is $SG = 0.65$. A pressure transducer is located in its side as illustrated. During testing of the vehicle, the tank is subjected to a constant linear acceleration, a_y.

FIND (a) Determine an expression that relates a_y and the pressure (in lb/ft^2) at the transducer. (b) What is the maximum acceleration that can occur before the fuel level drops below the transducer?



■ FIGURE E2.11

SOLUTION

(a) For a constant horizontal acceleration the fuel will move as a rigid body, and from Eq. 2.28 the slope of the fuel surface can be expressed as

$$\frac{dz}{dy} = -\frac{a_y}{g}$$

since $a_z = 0$. Thus, for some arbitrary a_y, the change in depth, z_1, of liquid on the right side of the tank can be found from the equation

$$-\frac{z_1}{0.75 \text{ ft}} = -\frac{a_y}{g}$$

or

$$z_1 = (0.75 \text{ ft})\left(\frac{a_y}{g}\right)$$

Since there is no acceleration in the vertical, z, direction, the pressure along the wall varies hydrostatically as shown by Eq. 2.26. Thus, the pressure at the transducer is given by the relationship

$$p = \gamma h$$

where h is the depth of fuel above the transducer, and therefore

$$p = (0.65)(62.4 \text{ lb/ft}^3)[0.5 \text{ ft} - (0.75 \text{ ft})(a_y/g)]$$

$$= 20.3 - 30.4\frac{a_y}{g} \quad \textbf{(Ans)}$$

for $z_1 \leq 0.5$ ft. As written, p would be given in lb/ft^2.

(b) The limiting value for $(a_y)_{max}$ (when the fuel level reaches the transducer) can be found from the equation

$$0.5 \text{ ft} = (0.75 \text{ ft})\left[\frac{(a_y)_{max}}{g}\right]$$

or

$$(a_y)_{max} = \frac{2g}{3}$$

and for standard acceleration of gravity

$$(a_y)_{max} = \tfrac{2}{3}(32.2 \text{ ft/s}^2) = 21.5 \text{ ft/s}^2 \quad \textbf{(Ans)}$$

COMMENT Note that the pressure in horizontal layers is not constant in this example since $\partial p/\partial y = -\rho a_y \neq 0$. Thus, for example, $p_1 \neq p_2$.

2.12.2 Rigid-Body Rotation

After an initial "start-up" transient, a fluid contained in a tank that rotates with a constant angular velocity ω about an axis as is shown in Fig. 2.30 will rotate with the tank as a rigid body. It is known from elementary particle dynamics that the acceleration of a fluid particle located at a distance r from the axis of rotation is equal in magnitude to $r\omega^2$, and the direction of the acceleration is toward the axis of rotation, as is illustrated in the figure. Since the paths of the fluid particles are circular, it is convenient to use cylindrical polar coordinates r, θ, and z, defined in the insert in Fig. 2.30. It will be shown in Chapter 6 that in terms of cylindrical coordinates the pressure gradient ∇p can be expressed as

$$\nabla p = \frac{\partial p}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial p}{\partial z}\hat{\mathbf{e}}_z \qquad \textbf{(2.29)}$$

Thus, in terms of this coordinate system

$$\mathbf{a}_r = -r\omega^2\,\hat{\mathbf{e}}_r \qquad \mathbf{a}_\theta = 0 \qquad \mathbf{a}_z = 0$$

and from Eq. 2.2

$$\frac{\partial p}{\partial r} = \rho r\omega^2 \qquad \frac{\partial p}{\partial \theta} = 0 \qquad \frac{\partial p}{\partial z} = -\gamma \qquad \textbf{(2.30)}$$

These results show that for this type of rigid-body rotation, the pressure is a function of two variables r and z, and therefore the differential pressure is

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$$

or

$$dp = \rho r\omega^2\,dr - \gamma\,dz \qquad \textbf{(2.31)}$$

On a horizontal plane ($dz = 0$), it follows from Eq. 2.31 that $dp/dr = \rho\omega^2 r$, which is greater than zero. Hence, as illustrated in the figure in the margin, because of centrifugal acceleration, the pressure increases in the radial direction.

Along a surface of constant pressure, such as the free surface, $dp = 0$, so that from Eq. 2.31 (using $\gamma = \rho g$)

$$\frac{dz}{dr} = \frac{r\omega^2}{g}$$

Integration of this result gives the equation for surfaces of constant pressure as

$$z = \frac{\omega^2 r^2}{2g} + \text{constant} \qquad \textbf{(2.32)}$$

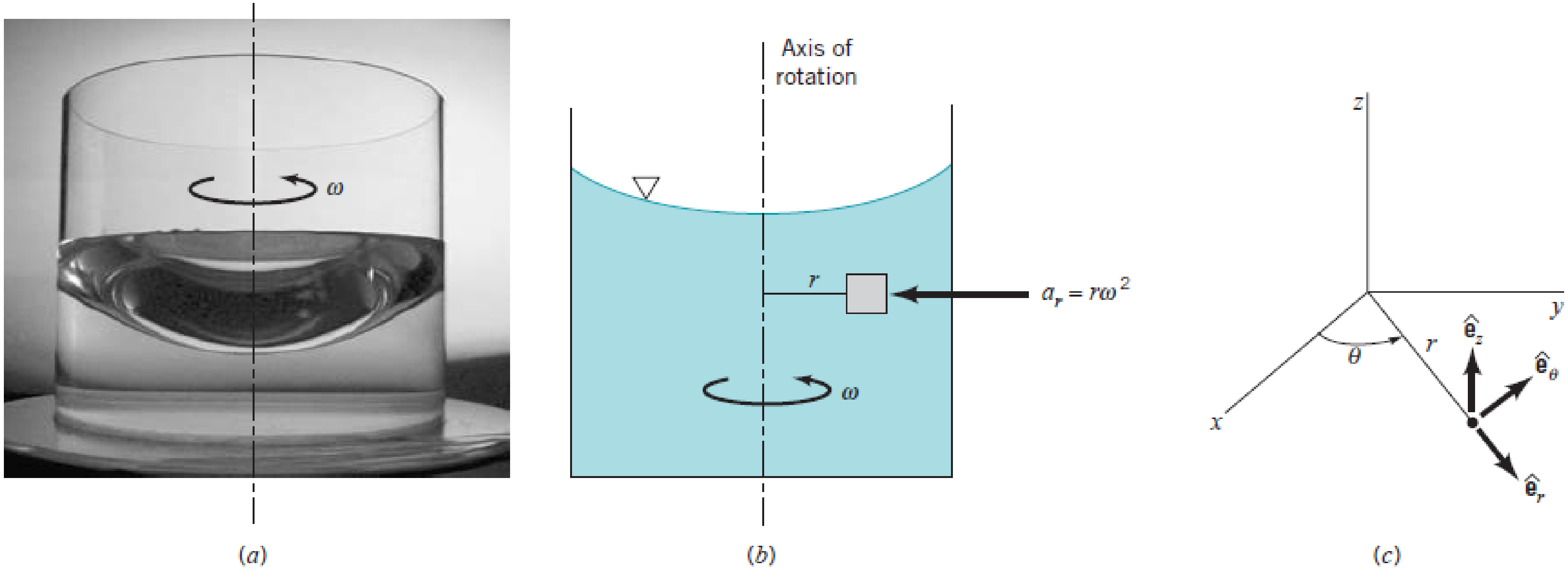
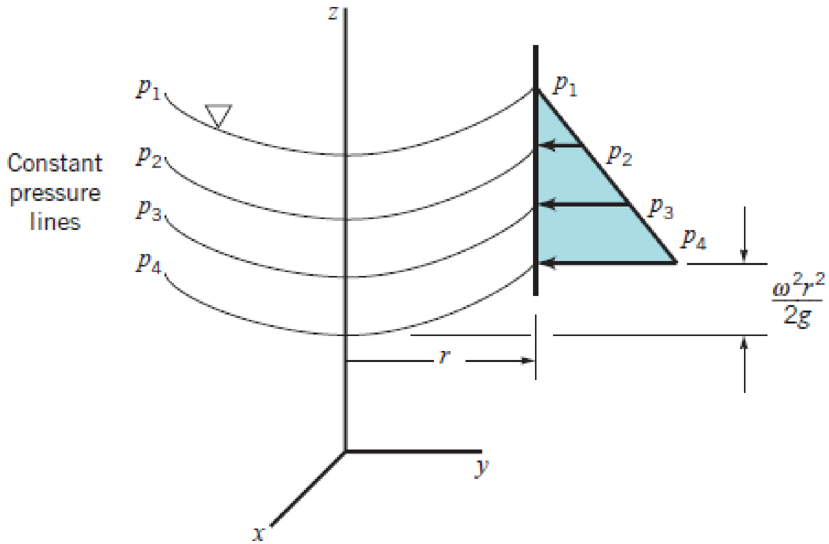


FIGURE 2.30 Rigid-body rotation of a liquid in a tank. (Photograph courtesy of Geno Pawlak.)



■ FIGURE 2.31 Pressure distribution in a rotating liquid.

This equation reveals that these surfaces of constant pressure are parabolic, as illustrated in Fig. 2.31.

Integration of Eq. 2.31 yields

$$\int dp = \rho\omega^2 \int r\,dr - \gamma \int dz$$

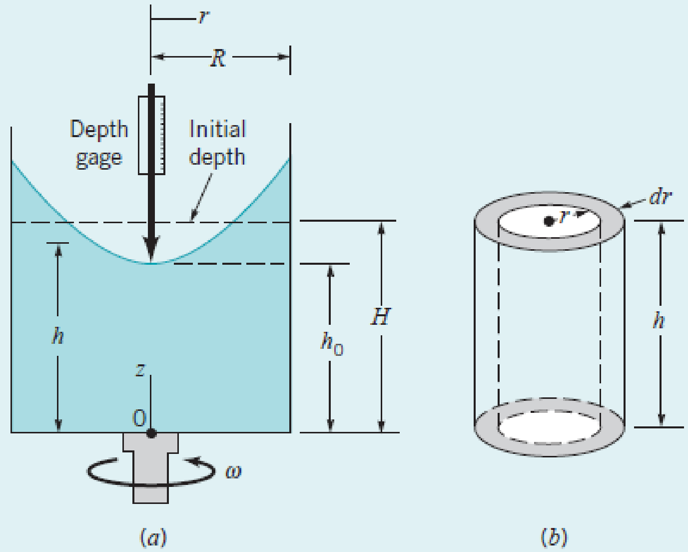
or

$$p = \frac{\rho\omega^2 r^2}{2} - \gamma z + \text{constant} \tag{2.33}$$

EXAMPLE 2.12 Free Surface Shape of Liquid in a Rotating Tank

GIVEN It has been suggested that the angular velocity, ω, of a rotating body or shaft can be measured by attaching an open cylinder of liquid, as shown in Fig. E2.12*a*, and measuring with some type of depth gage the change in the fluid level, $H - h_0$, caused by the rotation of the fluid.

FIND Determine the relationship between this change in fluid level and the angular velocity.



■ FIGURE E2.12

SOLUTION

The height, h, of the free surface above the tank bottom can be determined from Eq. 2.32, and it follows that

$$h = \frac{\omega^2 r^2}{2g} + h_0$$

The initial volume of fluid in the tank, $\forall_i$, is equal to

$$\forall_i = \pi R^2 H$$

The volume of the fluid with the rotating tank can be found with the aid of the differential element shown in Fig. E2.12*b*. This cylindrical shell is taken at some arbitrary radius, r, and its volume is

$$d\forall = 2\pi r h \, dr$$

The total volume is, therefore,

$$V\!\!\!\!/ = 2\pi \int_0^R r\left(\frac{\omega^2 r^2}{2g} + h_0\right) dr = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 h_0$$

Since the volume of the fluid in the tank must remain constant (assuming that none spills over the top), it follows that

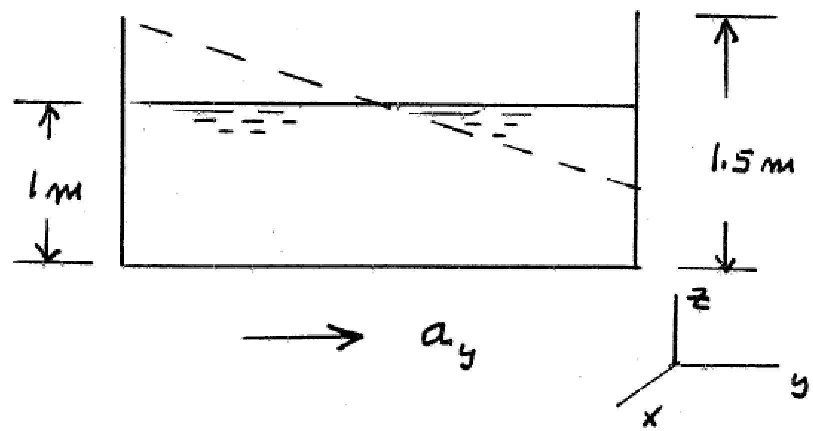
$$\pi R^2 H = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 h_0$$

or

$$H - h_0 = \frac{\omega^2 R^2}{4g} \qquad \textbf{(Ans)}$$

COMMENT This is the relationship we were looking for. It shows that the change in depth could indeed be used to determine the rotational speed, although the relationship between the change in depth and speed is not a linear one.

2.94 An open rectangular tank 1 m wide and 2 m long contains gasoline to a depth of 1 m. If the height of the tank sides is 1.5 m, what is the maximum horizontal acceleration (along the long axis of the tank) that can develop before the gasoline would begin to spill?



To prevent spilling,

$$\frac{dz}{dy} \geq -\frac{1.5\,m - 1.0\,m}{1\,m} = -0.50$$

(see figure).

Since,

$$\frac{dz}{dy} = -\frac{a_y}{g + a_z}$$

or, with $a_z = 0$,

$$a_y = -\left(\frac{dz}{dy}\right) g$$

so that

$$(a_y)_{max} = -(-0.50)(9.81\ \tfrac{m}{s^2}) = 4.91\ \tfrac{m}{s^2}$$

(Note: Acceleration could be either to the right or the left.)

Table .1 Centre of Gravity and Moment of Inertia for some typical shapes

	Shape	CG	I_G	I_{base}
1.	Triangle, side b height h and base zero of x axis	$h/3$	$bh^3/36$	$bh^3/12$
2.	Triangle, side b height h and vertex zero of x axis	$2h/3$	$bh^3/36$	$bh^3/12$
3.	Rectangle of width b and depth D	$D/2$	$bD^3/12$	$bD^3/3$
4.	Circle	$D/2$	$\pi D^4/64$	–
5.	Semicircle with diameter horizontal and zero of x axis	$2D/3\pi$	–	$\pi D^4/128$
6.	Quadrant of a circle, one radius horizontal	$4R/3\pi$	–	$\pi R^4/16$
7.	Ellipse : area $\pi bh/4$ Major axis is b, horizontal and minor axis is h	$h/2$	$\pi bh^3/64$	–
8.	Semi ellipse with major axis as horizontal and $x = 0$	$2h/3\pi$	–	$\pi bh^3/128$
9.	Parabola (half) area $2bh/3$ (from vertex as zero)	$y_g = 3h/5$ $x_g = 3b/8$	–	$2bh^3/7$

SUMMARY OF CHAPTER 2

Pressure gradient in a stationary fluid	$\frac{dp}{dz} = -\gamma$	**(2.4)**
Pressure variation in a stationary incompressible fluid	$p_1 = \gamma h + p_2$	**(2.7)**
Hydrostatic force on a plane surface	$F_R = \gamma h_c A$	**(2.18)**
Location of hydrostatic force on a plane surface	$y_R = \frac{I_{xc}}{y_c A} + y_c$	**(2.19)**
	$x_R = \frac{I_{xyc}}{y_c A} + x_c$	**(2.20)**
Buoyant force	$F_B = \gamma \forall$	**(2.22)**
Pressure gradient in rigid-body motion	$-\frac{\partial p}{\partial x} = \rho a_x, \quad -\frac{\partial p}{\partial y} = \rho a_y, \quad -\frac{\partial p}{\partial z} = \gamma + \rho a_z$	**(2.24)**
Pressure gradient in rigid-body rotation	$\frac{\partial p}{\partial r} = \rho r \omega^2, \quad \frac{\partial p}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} = -\gamma$	**(2.30)**