MENG353 - FLUID MECHANICS

SOURCE: FUNDAMENTALS OF FLUIDMECHANICS
MUNSON,P.GERHART,A.GERHART and HOCHSTEIN

CHAPTER 3 ELEMENTRY FLUID DYNAMICS

FALL 2017 - 18

ASSOC.PROF.DR.HASAN HACIŞEVKİ

EASTERN MEDITERRANEAN UNIVERSITY

Learning Objectives

Learning Objectives

After completing this chapter, you should be able to:

- discuss the application of Newton's second law to fluid flows.
- explain the development, uses, and limitations of the Bernoulli equation.
- use the Bernoulli equation (stand-alone or in combination with the simple continuity equation) to solve flow problems.
- apply the concepts of static, stagnation, dynamic, and total pressures.
- calculate various flow properties using the energy and hydraulic grade lines.

3.1 Newton's Second Law

Consider a really tiny volume of fluid, which is still large enough to contain a significant number of molecules. This volume is called a fluid particle (with further description in Section 4.1). As the fluid particle moves from one location to another, it usually experiences an acceleration or deceleration. According to Newton's second law of motion, the net force acting on the fluid particle under consideration must equal its mass times its acceleration,

$$\mathbf{F} = m\mathbf{a}$$

With the assumption of inviscid (or frictionless) flow, the fluid motion is governed by pressure and gravity forces only, and Newton's second law as it applies to a fluid particle is

(Net pressure force on particle) + (net gravity force on particle) =
(particle mass) × (particle acceleration)

The results of the interaction between the pressure, gravity, and acceleration provide numerous useful applications in fluid mechanics.

In this chapter we will be concerned with two-dimensional motion like that confined to the x–z plane as is shown in Fig. 3.1*a*. Clearly we could choose to describe the flow in terms of the components of acceleration and forces in the x and z coordinate directions. The resulting equations are frequently referred to as a two-dimensional form of the *Euler equations* of motion in rectangular Cartesian coordinates. This approach will be discussed in Chapter 6.

For steady flows each particle slides along its path, and its velocity vector is everywhere tangent to the path. The lines that are tangent to the velocity vectors throughout the flow field are called ***streamlines***. For many situations it is easiest to describe the flow in terms of the "streamline" coordinates based on the streamlines as are illustrated in Fig. 3.1*b*. The particle motion is described in terms of its distance, $s = s(t)$, along the streamline from some convenient origin and the local radius of curvature of the streamline, $\mathcal{R} = \mathcal{R}(s)$.

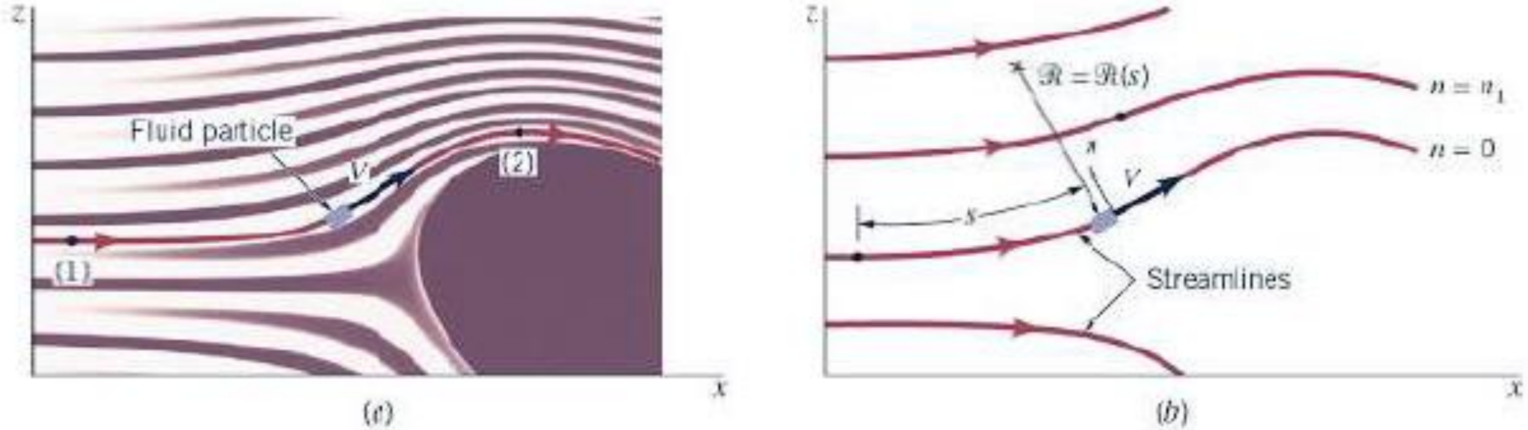


Figure 3.1 *(a)* **Flow in the x–z plane.** *(b)* **Flow in terms of streamline and normal coordinates.**

Streamwise acceleration results from two facts. First, the speed of the particle may change with time at a point, $V(t)$. Since we are limiting ourselves to steady flow, we will ignore velocity changes with time. (More information is in Section 3.8.2.) Second, the speed of the particle generally varies along the streamline, $V(s)$. For example, in Fig. 3.1a the speed may be 50 ft/s at point (1) and 100 ft/s at point (2). Thus, by use of the chain rule of differentiation, the s component of the acceleration is given by $a_s = dV/dt = (\partial V/\partial s)(ds/dt) = (\partial V/\partial s)V$. We have used the fact that speed is the time rate of change of distance, $V = ds/dt$. Note that the streamwise acceleration is the product of the rate of change of speed with distance along the streamline, $\partial V/\partial s$, and the speed, V. Since $\partial V/\partial s$ can be positive, negative, or zero, the streamwise acceleration can, therefore, be positive (acceleration), negative (deceleration), or zero (constant speed).

The normal component of acceleration, the centrifugal acceleration, is given in terms of the particle speed and the radius of curvature of its path. Thus, $a_n = V^2/\mathcal{R}$, where both V and $\mathcal{R}$ may vary along the streamline. These equations for the acceleration should be familiar from the study of particle motion in physics (Ref. 2) or dynamics (Ref. 1). A more complete derivation and discussion of these topics can be found in Chapter 4.

Thus, the (steady flow) components of acceleration in the s and n directions, a_s and a_n, are given by

$$a_s = V\frac{\partial V}{\partial s}, \qquad a_n = \frac{V^2}{\mathcal{R}} \tag{3.1}$$

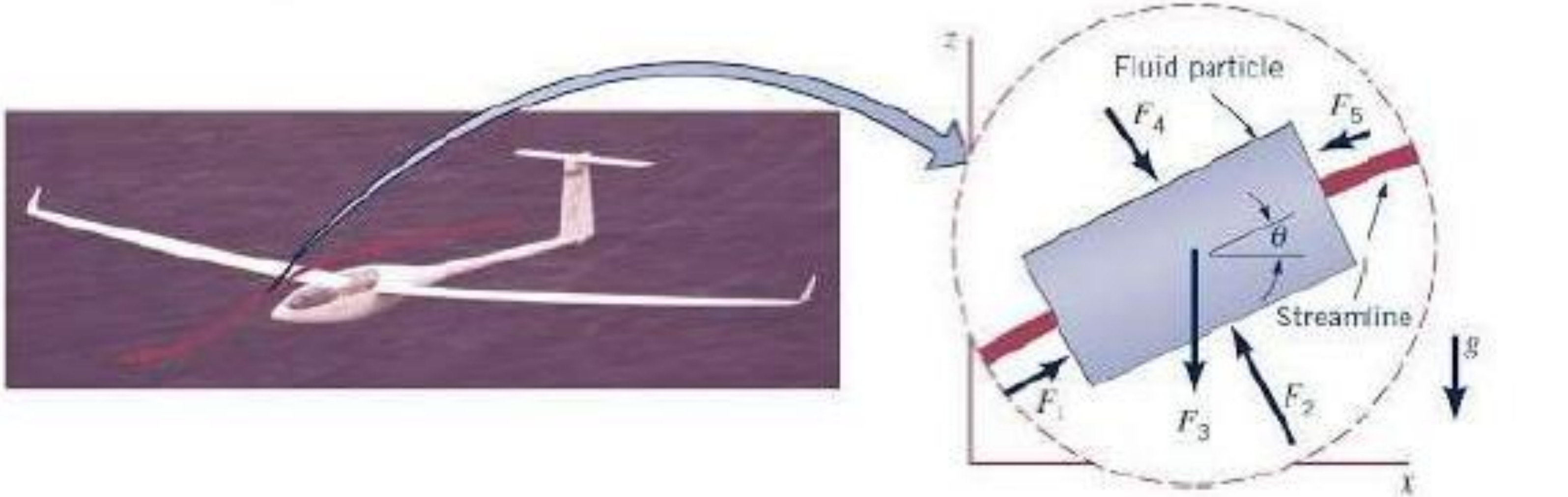


Figure 3.2 Isolation of a small fluid particle in a flow field. (Photo courtesy of Diana Sailplanes.)

3.2 F = *ma* along a Streamline

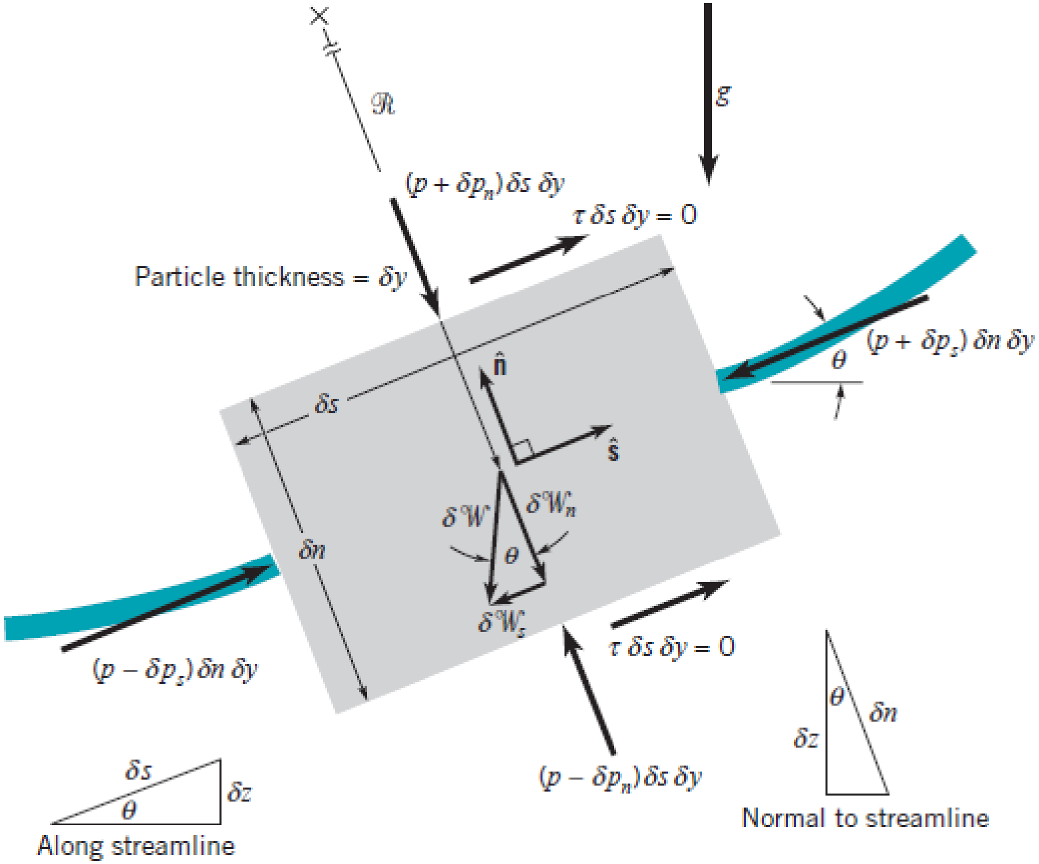
Consider the small fluid particle of size δs by δn in the plane of the figure and δy normal to the figure as shown in the free-body diagram of Fig. 3.3. Unit vectors along and normal to the streamline are denoted by $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$, respectively. For steady flow, the component of Newton's second law along the streamline direction, s, can be written as

$$\sum \delta F_s = \delta m\, a_s = \delta m\, V \frac{\partial V}{\partial s} = \rho\, \delta V\, V \frac{\partial V}{\partial s} \tag{3.2}$$

The gravity force (weight) on the particle can be written as $\delta W = \gamma\, \delta V$, where $\gamma = \rho g$ is the specific weight of the fluid (lb/ft^3 or N/m^3). Hence, the component of the weight force in the direction of the streamline is

$$\delta W_s = -\delta W \sin\theta = -\gamma\, \delta V \sin\theta$$

$$\delta p_s \approx \frac{\partial p}{\partial s}\frac{\delta s}{2}$$



■ **FIGURE 3.3** Free-body diagram of a fluid particle for which the important forces are those due to pressure and gravity.

Thus, if δF_{ps} is the net pressure force on the particle in the streamline direction, it follows that

$$\delta F_{ps} = (p - \delta p_s)\,\delta n\,\delta y - (p + \delta p_s)\,\delta n\,\delta y = -2\,\delta p_s\,\delta n\,\delta y$$
$$= -\frac{\partial p}{\partial s}\delta s\,\delta n\,\delta y = -\frac{\partial p}{\partial s}\delta V\!\!\!\!-$$

Note that the actual level of the pressure, p, is not important. What produces a net pressure force is the fact that the pressure is not constant throughout the fluid. The nonzero pressure gradient, $\nabla p = \partial p/\partial s\,\hat{\mathbf{s}} + \partial p/\partial n\,\hat{\mathbf{n}}$, is what provides a net pressure force on the particle. Viscous forces, represented by $\tau\,\delta s\,\delta y$, are zero, since the fluid is inviscid.

Thus, the net force acting in the streamline direction on the particle shown in Fig. 3.3 is given by

$$\sum \delta F_s = \delta \mathcal{W}_s + \delta F_{ps} = \left(-\gamma \sin\theta - \frac{\partial p}{\partial s}\right)\delta V\!\!\!\!- \qquad \textbf{(3.3)}$$

By combining Eqs. 3.2 and 3.3, we obtain the following equation of motion along the streamline direction:

$$-\gamma \sin\theta - \frac{\partial p}{\partial s} = \rho V \frac{\partial V}{\partial s} = \rho a_s \qquad \textbf{(3.4)}$$

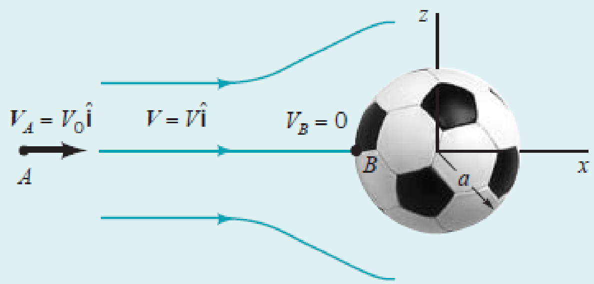
EXAMPLE 3.1 Pressure Variation along a Streamline

GIVEN Consider the inviscid, incompressible, steady flow along the horizontal streamline A–B in front of the sphere of radius a, as shown in Fig. E3.1a. From a more advanced theory of flow past a sphere, the fluid velocity along this streamline is

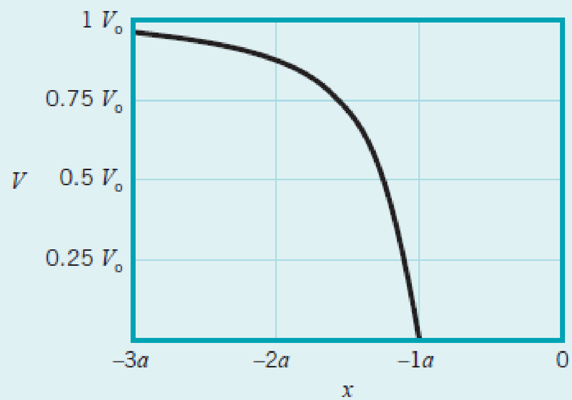
$$V = V_0\left(1 + \frac{a^3}{x^3}\right)$$

as shown in Fig. E3.1b.

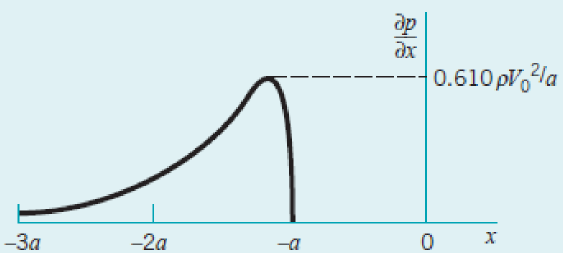
FIND Determine the pressure variation along the streamline from point A far in front of the sphere ($x_A = -\infty$ and $V_A = V_0$) to point B on the sphere ($x_B = -a$ and $V_B = 0$).



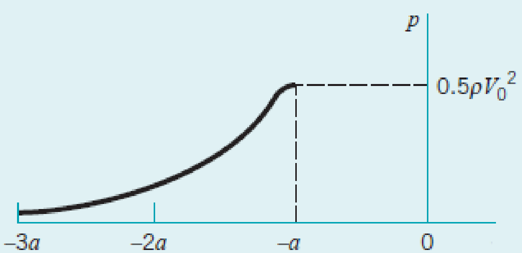
(a)



(b)



(c)



(d)

■ FIGURE E3.1

Solution

Since the flow is steady and inviscid, Eq. 3.4 is valid. In addition, since the streamline is horizontal, $\sin\theta = \sin 0° = 0$ and the equation of motion along the streamline reduces to

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} \tag{1}$$

With the given velocity variation along the streamline, the acceleration term is

$$V\frac{\partial V}{\partial s} = V\frac{\partial V}{\partial x} = V_0\left(1 + \frac{a^3}{x^3}\right)\left(-\frac{3V_0a^3}{x^4}\right)$$
$$= -3V_0^2\left(1 + \frac{a^3}{x^3}\right)\frac{a^3}{x^4}$$

where we have replaced s by x since the two coordinates are identical (within an additive constant) along streamline A–B. It follows that $V\,\partial V/\partial s < 0$ along the streamline. The fluid slows down from V_0 far ahead of the sphere to zero velocity on the "nose" of the sphere ($x = -a$).

Thus, according to Eq. 1, to produce the given motion the pressure gradient along the streamline is

$$\frac{\partial p}{\partial x} = \frac{3\rho a^3 V_0^2 (1 + a^3/x^3)}{x^4} \tag{2}$$

This variation is indicated in Fig. E3.1c. It is seen that the pressure increases in the direction of flow ($\partial p/\partial x > 0$) from point A to point B. The maximum pressure gradient ($0.610\ \rho V_0^2/a$) occurs just slightly ahead of the sphere ($x = -1.205a$). It is the pressure gradient that slows the fluid down from $V_A = V_0$ to $V_B = 0$ as shown in Fig. E3.1b.

The pressure distribution along the streamline can be obtained by integrating Eq. 2 from $p = 0$ (gage) at $x = -\infty$ to pressure p at location x. The result, plotted in Fig. E3.1d, is

$$p = -\rho V_0^2\left[\left(\frac{a}{x}\right)^3 + \frac{(a/x)^6}{2}\right] \qquad \textbf{(Ans)}$$

COMMENT The pressure at B, a stagnation point since $V_B = 0$, is the highest pressure along the streamline ($p_B = \rho V_0^2/2$). As shown in Chapter 9, this excess pressure on the front of the sphere (i.e., $p_B > 0$) contributes to the net drag force on the sphere. Note that the pressure gradient and pressure are directly proportional to the density of the fluid, a representation of the fact that the fluid inertia is proportional to its mass.

Equation 3.4 can be rearranged and integrated as follows. First, we note from Fig. 3.3 that along the streamline $\sin\theta = dz/ds$. Also, we can write $V\,dV/ds = \frac{1}{2}d(V^2)/ds$. Finally, along the streamline the value of n is constant ($dn = 0$) so that $dp = (\partial p/\partial s)\,ds + (\partial p/\partial n)\,dn = (\partial p/\partial s)\,ds$. Hence, as indicated by the figure in the margin, along a given streamline $p(s, n) = p(s)$ and $\partial p/\partial s = dp/ds$. These ideas combined with Eq. 3.4 give the following result valid along a streamline

$$-\gamma \frac{dz}{ds} - \frac{dp}{ds} = \frac{1}{2}\rho \frac{d(V^2)}{ds}$$

This simplifies to

$$dp + \frac{1}{2}\rho d(V^2) + \gamma\, dz = 0 \qquad \text{(along a streamline)} \qquad \textbf{(3.5)}$$

which, for constant acceleration of gravity, can be integrated to give

$$\int \frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C \qquad \text{(along a streamline)} \qquad \textbf{(3.6)}$$

where C is a constant of integration to be determined by the conditions at some point on the streamline.

With the additional assumption that the density remains constant (a very good assumption for liquids and also for gases if the speed is "not too high"), Eq. 3.6 assumes the following simple representation for steady, inviscid, incompressible flow.

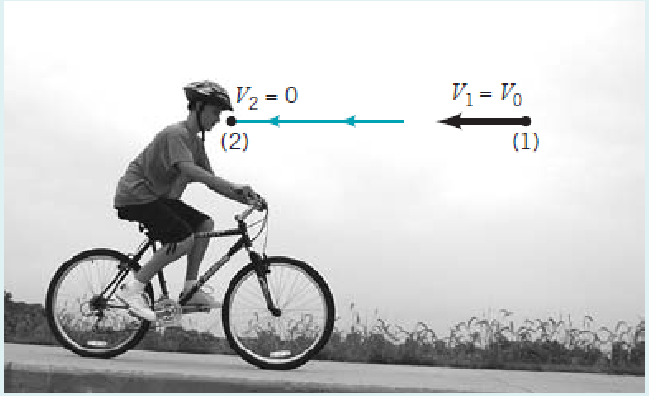
$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline} \tag{3.7}$$

This is the celebrated ***Bernoulli equation***—a very powerful tool in fluid mechanics. In 1738 Daniel Bernoulli (1700–1782) published his *Hydrodynamics* in which an equivalent of this famous equation first appeared. To use it correctly we must constantly remember the basic assumptions used in its derivation: (1) viscous effects are assumed negligible, (2) the flow is assumed to be steady, (3) the flow is assumed to be incompressible, (4) the equation is applicable along a streamline. In the derivation of Eq. 3.7, we assume that the flow takes place in a plane (the x–z plane). In general, this equation is valid for both planar and nonplanar (three-dimensional) flows, provided it is applied along the streamline.

EXAMPLE 3.2 The Bernoulli Equation

GIVEN Consider the flow of air around a bicyclist moving through still air with velocity V_0, as is shown in Fig. E3.2.

FIND Determine the difference in the pressure between points (1) and (2).



■ FIGURE E3.2

SOLUTION

In a coordinate fixed to the ground, the flow is unsteady as the bicyclist rides by. However, in a coordinate system fixed to the bike, it appears as though the air is flowing steadily toward the bicyclist with speed V_0. Since use of the Bernoulli equation is restricted to steady flows, we select the coordinate system fixed to the bike. If the assumptions of Bernoulli's equation are valid (steady, incompressible, inviscid flow), Eq. 3.7 can be applied as follows along the streamline that passes through (1) and (2)

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

We consider (1) to be in the free stream so that $V_1 = V_0$ and (2) to be at the tip of the bicyclist's nose and assume that $z_1 = z_2$ and $V_2 = 0$ (both of which, as is discussed in Section 3.4, are reasonable assumptions). It follows that the pressure at (2) is greater than that at (1) by an amount

$$p_2 - p_1 = \tfrac{1}{2}\rho V_1^2 = \tfrac{1}{2}\rho V_0^2 \qquad \text{(Ans)}$$

COMMENTS A similar result was obtained in Example 3.1 by integrating the pressure gradient, which was known because the velocity distribution along the streamline, $V(s)$, was known. The Bernoulli equation is a general integration of $\mathbf{F} = m\mathbf{a}$. To determine $p_2 - p_1$, knowledge of the detailed velocity distribution is not needed—only the "boundary conditions" at (1) and (2) are required. Of course, knowledge of the value of V along the streamline is needed to determine the pressure at points between (1) and (2). Note that if we measure $p_2 - p_1$ we can determine the speed, V_0. As discussed in Section 3.5, this is the principle upon which many velocity measuring devices are based.

If the bicyclist were accelerating or decelerating, the flow would be unsteady (i.e., $V_0 \neq$ constant) and the above analysis would be incorrect since Eq. 3.7 is restricted to steady flow.

3.3 F = *ma* Normal to a Streamline

We again consider the force balance on the fluid particle shown in Fig. 3.3 and the figure in the margin. This time, however, we consider components in the normal direction, $\hat{\mathbf{n}}$, and write Newton's second law in this direction as

$$\sum \delta F_n = \frac{\delta m\, V^2}{\mathcal{R}} = \frac{\rho\, \delta \forall\, V^2}{\mathcal{R}} \tag{3.8}$$

where $\sum \delta F_n$ represents the sum of n components of all the forces acting on the particle and δm is particle mass. We assume the flow is steady with a normal acceleration $a_n = V^2/\mathcal{R}$, where $\mathcal{R}$ is the local radius of curvature of the streamlines. This acceleration is produced by the change in direction of the particle's velocity as it moves along a curved path.

We again assume that the only forces of importance are pressure and gravity. The component of the weight (gravity force) in the normal direction is

$$\delta \mathcal{W}_n = -\delta \mathcal{W} \cos\theta = -\gamma\, \delta \forall \cos\theta$$

If the streamline is vertical at the point of interest, $\theta = 90°$, and there is no component of the particle weight normal to the direction of flow to contribute to its acceleration in that direction.

If the pressure at the center of the particle is p, then its values on the top and bottom of the particle are $p + \delta p_n$ and $p - \delta p_n$, where $\delta p_n = (\partial p/\partial n)(\delta n/2)$. Thus, if δF_{pn} is the net pressure force on the particle in the normal direction, it follows that

$$\delta F_{pn} = (p - \delta p_n)\,\delta s\,\delta y - (p + \delta p_n)\,\delta s\,\delta y = -2\,\delta p_n\,\delta s\,\delta y$$

$$= -\frac{\partial p}{\partial n}\delta s\,\delta n\,\delta y = -\frac{\partial p}{\partial n}\delta \forall$$

Hence, the net force acting in the normal direction on the particle shown in Fig 3.3 is given by

$$\sum \delta F_n = \delta \mathcal{W}_n + \delta F_{pn} = \left(-\gamma \cos\theta - \frac{\partial p}{\partial n}\right)\delta V\!\!\!\!/ \quad \textbf{(3.9)}$$

By combining Eqs. 3.8 and 3.9 and using the fact that along a line normal to the streamline $\cos\theta = dz/dn$ (see Fig. 3.3), we obtain the following equation of motion along the normal direction

$$-\gamma \frac{dz}{dn} - \frac{\partial p}{\partial n} = \frac{\rho V^2}{\mathcal{R}} \quad \textbf{(3.10a)}$$

The physical interpretation of Eq. 3.10 is that a change in the direction of flow of a fluid particle (i.e., a curved path, $\mathcal{R} < \infty$) is accomplished by the appropriate combination of pressure gradient and particle weight normal to the streamline. A larger speed or density or a smaller radius of curvature of the motion requires a larger force unbalance to produce the motion. For example, if gravity is neglected (as is commonly done for gas flows) or if the flow is in a horizontal ($dz/dn = 0$) plane, Eq. 3.10 becomes

$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathcal{R}} \quad \textbf{(3.10b)}$$

EXAMPLE 3.3 Pressure Variation Normal to a Streamline

GIVEN Shown in Figs. E3.3*a,b* are two flow fields with circular streamlines. The velocity distributions are

$$V(r) = (V_0/r_0)r \quad \text{for case } (a)$$

and

$$V(r) = \frac{(V_0 r_0)}{r} \quad \text{for case } (b)$$

where V_0 is the velocity at $r = r_0$.

FIND Determine the pressure distributions, $p = p(r)$, for each, given that $p = p_0$ at $r = r_0$.

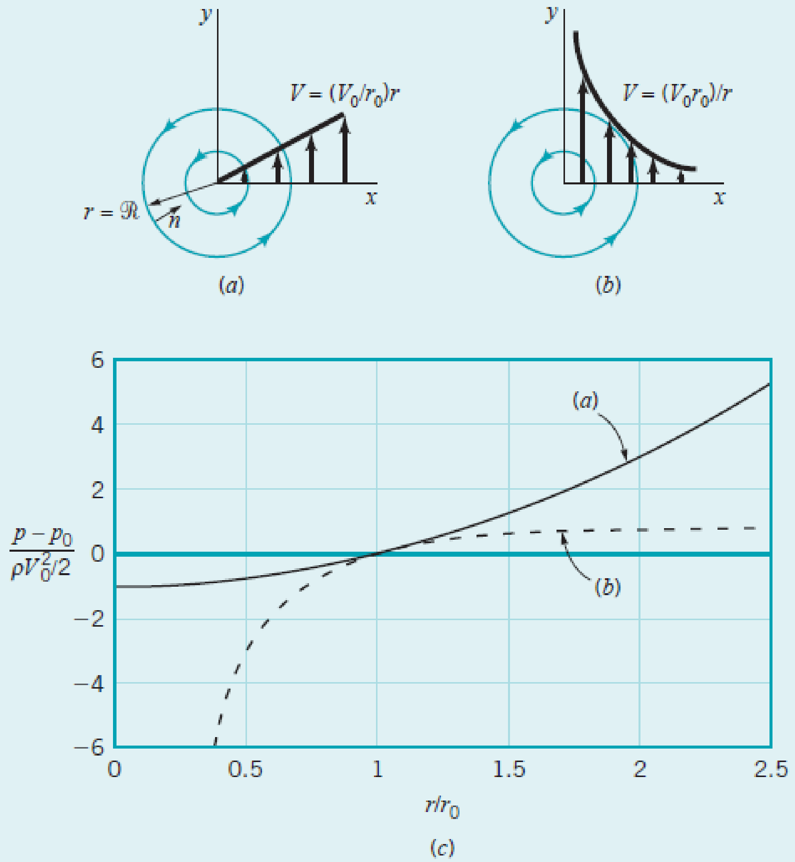
SOLUTION

We assume the flows are steady, inviscid, and incompressible with streamlines in the horizontal plane ($dz/dn = 0$). Because the streamlines are circles, the coordinate n points in a direction opposite that of the radial coordinate, $\partial/\partial n = -\partial/\partial r$, and the radius of curvature is given by $\mathcal{R} = r$. Hence, Eq. 3.9 becomes

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r}$$

For case (a) this gives

$$\frac{\partial p}{\partial r} = \rho(V_0/r_0)^2 r$$



■ FIGURE E3.3

whereas for case (b) it gives

$$\frac{\partial p}{\partial r} = \frac{\rho (V_0 r_0)^2}{r^3}$$

For either case the pressure increases as r increases since $\partial p/\partial r > 0$. Integration of these equations with respect to r, starting with a known pressure $p = p_0$ at $r = r_0$, gives

$$p - p_0 = (\rho V_0^2/2)[(r/r_0)^2 - 1] \qquad \textbf{(Ans)}$$

for case (a) and

$$p - p_0 = (\rho V_0^2/2)[1 - (r_0/r)^2] \qquad \textbf{(Ans)}$$

for case (b). These pressure distributions are shown in Fig. E3.3*c*.

COMMENT The pressure distributions needed to balance the centrifugal accelerations in cases (a) and (b) are not the same because the velocity distributions are different. In fact, for case (a) the pressure increases without bound as $r \to \infty$, whereas for case (b) the pressure approaches a finite value as $r \to \infty$. The streamline patterns are the same for each case, however.

Physically, case (a) represents rigid body rotation (as obtained in a can of water on a turntable after it has been "spun up") and case (b) represents a free vortex (an approximation to a tornado, a hurricane, or the swirl of water in a drain, the "bathtub vortex"). See Fig. E6.6 for an approximation of this type of flow.

If we multiply Eq. 3.10 by dn, use the fact that $\partial p/\partial n = dp/dn$ if s is constant, and integrate across the streamline (in the n direction) we obtain

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\mathcal{R}} dn + gz = \text{constant across the streamline} \qquad \textbf{(3.11)}$$

Thus, the final form of Newton's second law applied across the streamlines for steady, inviscid, incompressible flow is

$$\boxed{p + \rho \int \frac{V^2}{\mathcal{R}} dn + \gamma z = \text{constant across the streamline}} \qquad \textbf{(3.12)}$$

As with the Bernoulli equation, we must be careful that the assumptions involved in the derivation of this equation are not violated when it is used.

3.4 Physical Interpretation

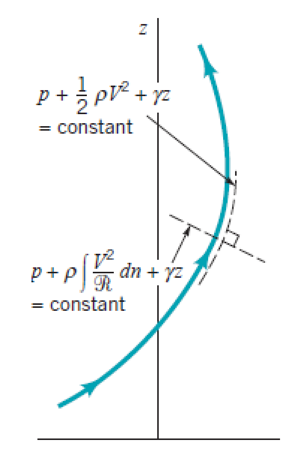
In the previous two sections, we developed the basic equations governing fluid motion under a fairly stringent set of restrictions. In spite of the numerous assumptions imposed on these flows, a variety of flows can be readily analyzed with them. A physical interpretation of the equations will be of help in understanding the processes involved. To this end, we rewrite Eqs. 3.7 and 3.12 here and interpret them physically. Application of $\mathbf{F} = m\mathbf{a}$ along and normal to the streamline results in

$$p + \tfrac{1}{2}\rho V^2 + \gamma z = \text{constant along the streamline} \tag{3.13}$$

and

$$p + \rho \int \frac{V^2}{\mathcal{R}} dn + \gamma z = \text{constant across the streamline} \tag{3.14}$$

as indicated by the figure in the margin.



The following basic assumptions were made to obtain these equations: The flow is steady and the fluid is inviscid and incompressible. In practice none of these assumptions is exactly true.

An alternate but equivalent form of the Bernoulli equation is obtained by dividing each term of Eq. 3.7 by the specific weight, γ, to obtain

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline}$$

The elevation term, z, is related to the potential energy of the particle and is called the ***elevation head***. The pressure term, p/γ, is called the ***pressure head*** and represents the height of a column of the fluid that is needed to produce the pressure p. The velocity term, $V^2/2g$, is the ***velocity head*** and represents the vertical distance needed for the fluid to fall freely (neglecting friction) if it is to reach velocity V from rest. The Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline.

3.5 Static, Stagnation, Dynamic, and Total Pressure

A useful concept associated with the Bernoulli equation deals with the stagnation and dynamic pressures. These pressures arise from the conversion of kinetic energy in a flowing fluid into a "pressure rise" as the fluid is brought to rest (as in Example 3.2). In this section we explore various results of this process. Each term of the Bernoulli equation, Eq. 3.13, has the dimensions of force per unit area—psi, lb/ft^2, N/m^2.

$$p + \frac{1}{2}\rho V^2 + \gamma z = p_T = \text{constant along a streamline} \qquad \textbf{(3.15)}$$

Again, we must be careful that the assumptions used in the derivation of this equation are appropriate for the flow being considered.

The Bernoulli equation is a statement that the total pressure remains constant along a streamline.

Knowledge of the values of the static and stagnation pressures in a fluid implies that the fluid speed can be calculated. This is the principle on which the ***Pitot-static tube*** is based [H. de Pitot (1695–1771)]. As shown in Fig. 3.6, two concentric tubes are attached to two pressure gages (or a differential gage) so that the values of p_3 and p_4 (or the difference $p_3 - p_4$) can be determined. The center tube measures the stagnation pressure at its open tip. If elevation changes are negligible,

$$p_3 = p + \tfrac{1}{2}\rho V^2$$

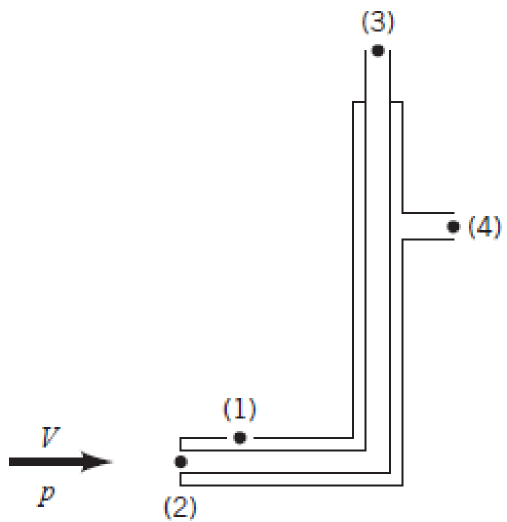


FIGURE 3.6 The Pitot-static tube.

where p and V are the pressure and velocity of the fluid upstream of point (2). The outer tube is made with several small holes at an appropriate distance from the tip so that they measure the static pressure. If the effect of the elevation difference between (1) and (4) is negligible, then

$$p_4 = p_1 = p$$

By combining these two equations we see that

$$p_3 - p_4 = \tfrac{1}{2}\rho V^2$$

which can be rearranged to give

$$V = \sqrt{2(p_3 - p_4)/\rho} \tag{3.16}$$

EXAMPLE 3.6 Pitot-Static Tube

GIVEN An airplane flies 200 mi/hr at an elevation of 10,000 ft in a standard atmosphere as shown in Fig. E3.6*a*.

FIND Determine the pressure at point (1) far ahead of the airplane, the pressure at the stagnation point on the nose of the airplane, point (2), and the pressure difference indicated by a Pitot-static probe attached to the fuselage.

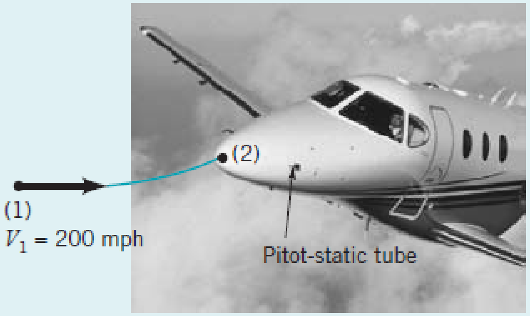


FIGURE E3.6*a* (Photo courtesy of Hawker Beechcraft.)

SOLUTION

From Table C.1 we find that the static pressure at the altitude given is

$$p_1 = 1456 \text{ lb/ft}^2 \text{ (abs)} = 10.11 \text{ psia} \qquad \textbf{(Ans)}$$

Also, the density is $\rho = 0.001756 \text{ slug/ft}^3$.

If the flow is steady, inviscid, and incompressible and elevation changes are neglected, Eq. 3.13 becomes

$$p_2 = p_1 + \frac{\rho V_1^2}{2}$$

With $V_1 = 200 \text{ mi/hr} = 293 \text{ ft/s}$ and $V_2 = 0$ (since the coordinate system is fixed to the airplane) we obtain

$$p_2 = 1456 \text{ lb/ft}^2 + (0.001756 \text{ slugs/ft}^3)(293^2 \text{ ft}^2/\text{s}^2)/2$$

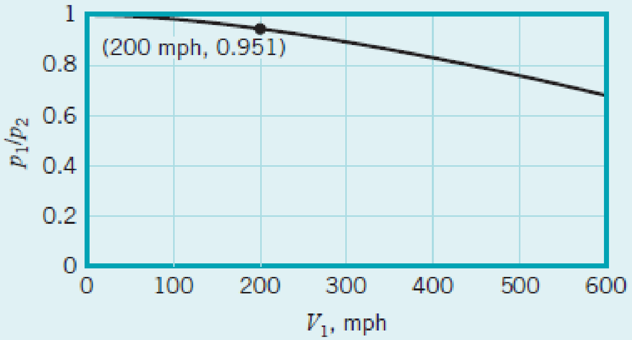


FIGURE E3.6*b*

$$= (1456 + 75.4) \text{ lb/ft}^2 \text{ (abs)}$$

Hence, in terms of gage pressure

$$p_2 = 75.4 \text{ lb/ft}^2 = 0.524 \text{ psi} \qquad \textbf{(Ans)}$$

Thus, the pressure difference indicated by the Pitot-static tube is

$$p_2 - p_1 = \frac{\rho V_1^2}{2} = 0.524 \text{ psi} \qquad \textbf{(Ans)}$$

COMMENTS Note that it is very easy to obtain incorrect results by using improper units. Do not add lb/in.^{2} and lb/ft^2. Recall that $(\text{slug/ft}^3)(\text{ft}^2/\text{s}^2) = (\text{slug} \cdot \text{ft/s}^2)/(\text{ft}^2) = \text{lb/ft}^2$.

It was assumed that the flow is incompressible—the density remains constant from (1) to (2). However, since $\rho = p/RT$, a change in pressure (or temperature) will cause a change in density. For this relatively low speed, the ratio of the absolute pressures is nearly unity [i.e., $p_1/p_2 = (10.11 \text{ psia})/(10.11 + 0.524 \text{ psia}) = 0.951$], so that the density change is negligible. However, by repeating the calculations for various values of the speed, V_1, the results shown in Fig. E3.6*b* are obtained. Clearly at the 500 to 600 mph speeds normally flown by commercial airliners, the pressure ratio is such that density changes are important. In such situations it is necessary to use compressible flow concepts to obtain accurate results. (See Section 3.8.1 and Chapter 11.)

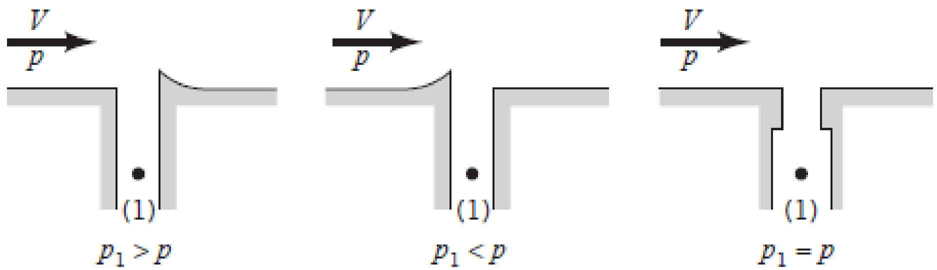


FIGURE 3.8 Incorrect and correct design of static pressure taps.

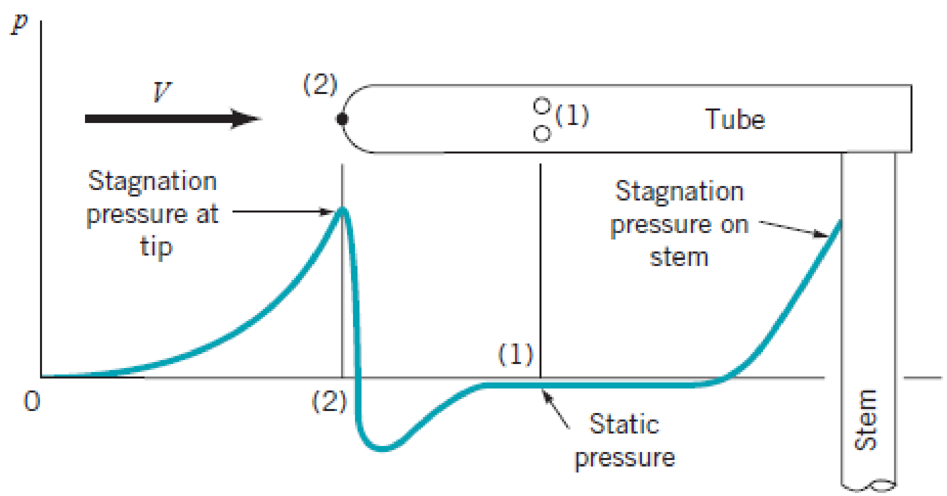


FIGURE 3.9 Typical pressure distribution along a Pitot-static tube.

3.6 Examples of Use of the Bernoulli Equation

In this section we illustrate various additional applications of the Bernoulli equation. Between any two points, (1) and (2), on a streamline in steady, inviscid, incompressible flow the Bernoulli equation can be applied in the form

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \quad \mathbf{(3.17)}$$

Obviously if five of the six variables are known, the remaining one can be determined. In many instances it is necessary to introduce other equations, such as the conservation of mass. Such considerations will be discussed briefly in this section and in more detail in Chapter 5.

3.6.1 Free Jets

One of the oldest equations in fluid mechanics deals with the flow of a liquid from a large reservoir. A modern version of this type of flow involves the flow of coffee from a coffee urn as indicated by the figure in the margin. The basic principles of this type of flow are shown in Fig. 3.11 where a jet of liquid of diameter d flows from the nozzle with velocity V. (A nozzle is a device shaped to accelerate a fluid.) Application of Eq. 3.17 between points (1) and (2) on the streamline shown gives

$$\gamma h = \tfrac{1}{2}\rho V^2$$

We have used the facts that $z_1 = h$, $z_2 = 0$, the reservoir is large ($V_1 \cong 0$) and open to the atmosphere ($p_1 = 0$ gage), and the fluid leaves as a ***"free jet"*** ($p_2 = 0$). Thus, we obtain

$$V = \sqrt{2\frac{\gamma h}{\rho}} = \sqrt{2gh} \qquad \textbf{(3.18)}$$

which is the modern version of a result obtained in 1643 by Torricelli (1608–1647), an Italian physicist.

The fact that the exit pressure equals the surrounding pressure ($p_2 = 0$) can be seen by applying $\mathbf{F} = m\mathbf{a}$, as given by Eq. 3.14, across the streamlines between (2) and (4). If the streamlines at the tip of the nozzle are straight ($\mathscr{R} = \infty$), it follows that $p_2 = p_4$. Since (4) is on the surface of the jet, in contact with the atmosphere, we have $p_4 = 0$. Thus, $p_2 = 0$ also. Since (2) is an arbitrary point in the exit plane of the nozzle, it follows that the pressure is atmospheric across this plane. Physically, since there is no component of the weight force or acceleration in the normal (horizontal) direction, the pressure is constant in that direction.

Once outside the nozzle, the stream continues to fall as a free jet with zero pressure throughout ($p_5 = 0$) and as seen by applying Eq. 3.17 between points (1) and (5), the speed increases according to

$$V = \sqrt{2g\,(h + H)}$$

where H is the distance the fluid has fallen outside the nozzle.

Equation 3.18 could also be obtained by writing the Bernoulli equation between points (3) and (4) using the fact that $z_4 = 0$, $z_3 = \ell$. Also, $V_3 = 0$ since it is far from the nozzle, and from hydrostatics, $p_3 = \gamma(h - \ell)$.

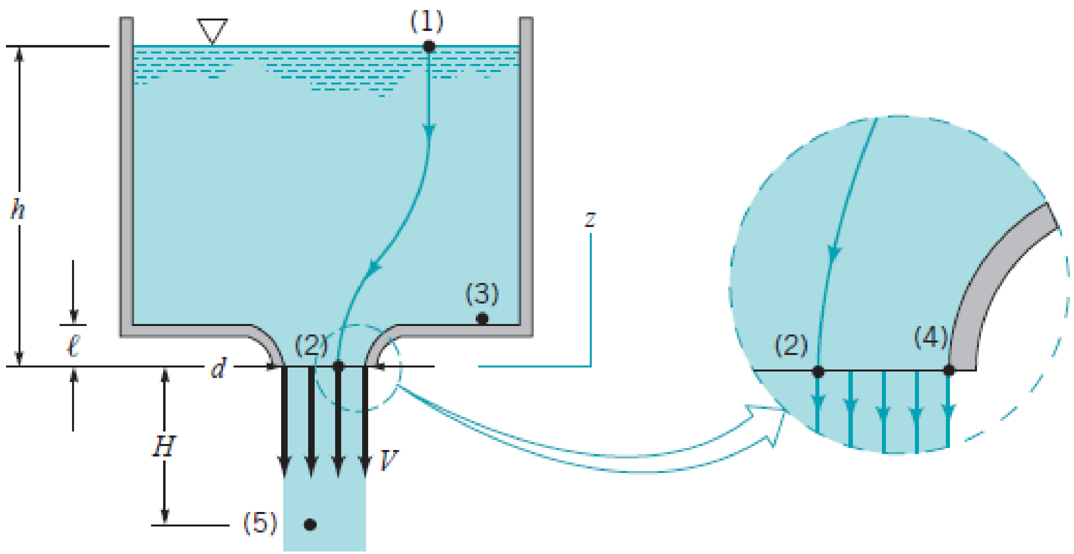


FIGURE 3.11 Vertical flow from a tank.

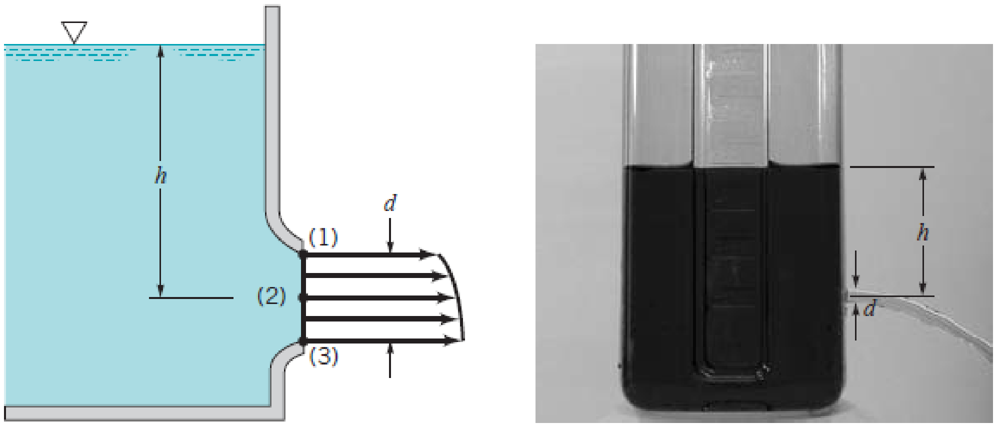


FIGURE 3.12 Horizontal flow from a tank.

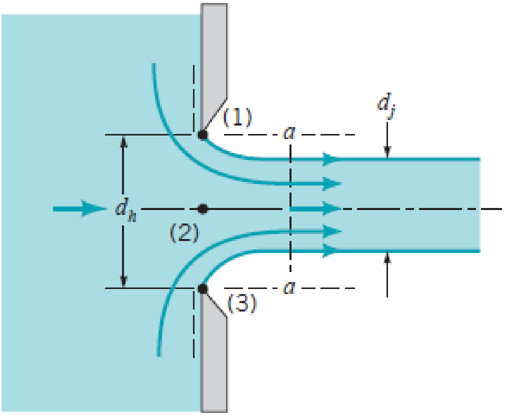


FIGURE 3.13 Vena contracta effect for a sharp-edged orifice.

For the horizontal nozzle of Fig. 3.12*a*, the velocity of the fluid at the centerline, V_2,will be slightly greater than that at the top, V_1, and slightly less than that at the bottom, V_3, due to the differences in elevation. In general, $d \ll h$ as shown in Fig. 3.12*b* and we can safely use the centerline velocity as a reasonable "average velocity."

If the exit is not a smooth, well-contoured nozzle, but rather a flat plate as shown in Fig. 3.13, the diameter of the jet, d_j, will be less than the diameter of the hole, d_h. This phenomenon, called a *vena contracta* effect, is a result of the inability of the fluid to turn the sharp 90° corner indicated by the dotted lines in the figure.

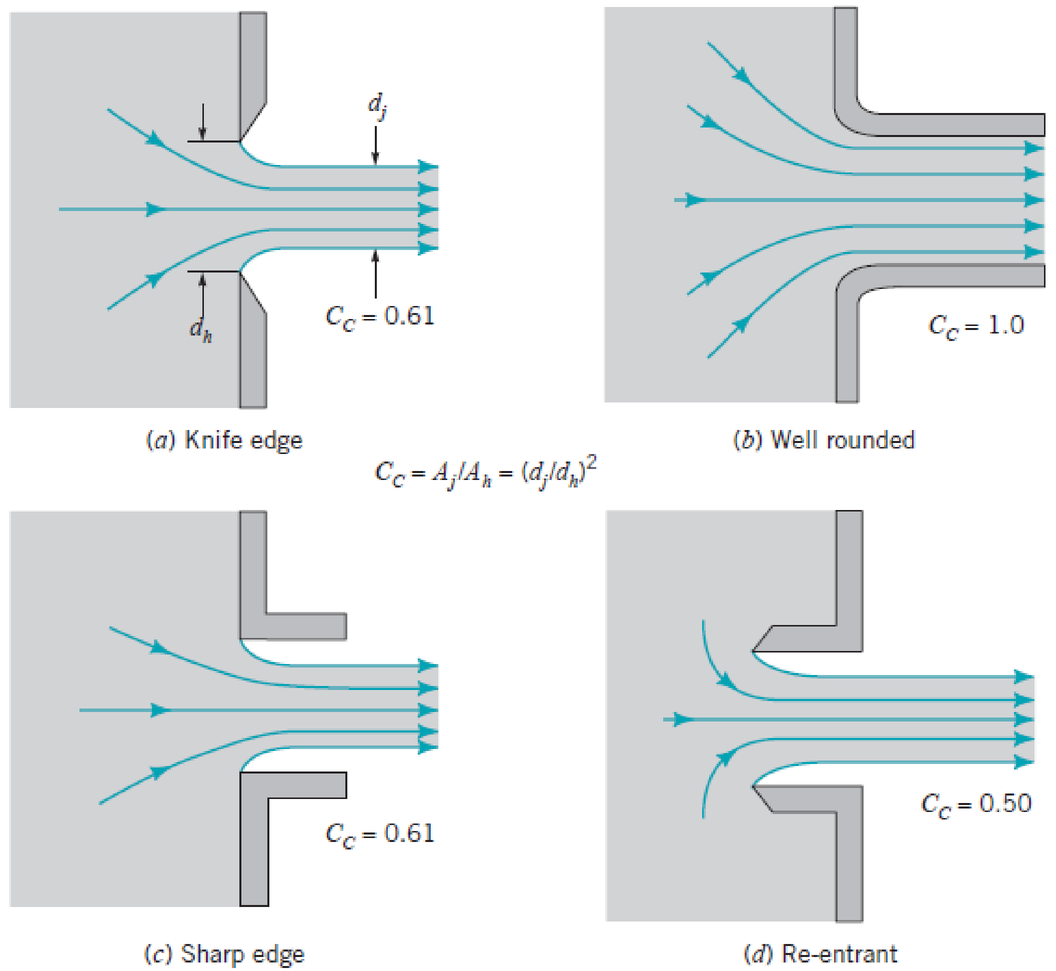
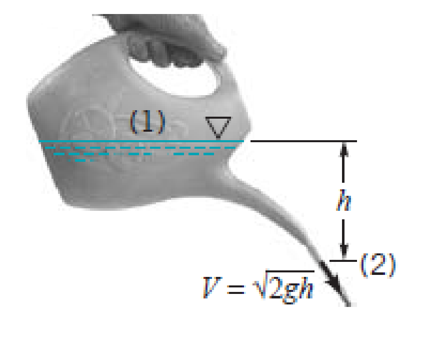


FIGURE 3.14 Typical flow patterns and contraction coefficients for various round exit configurations. (*a*) Knife edge, (*b*) Well rounded, (*c*) Sharp edge, (*d*) Re-entrant.



3.6.2 Confined Flows

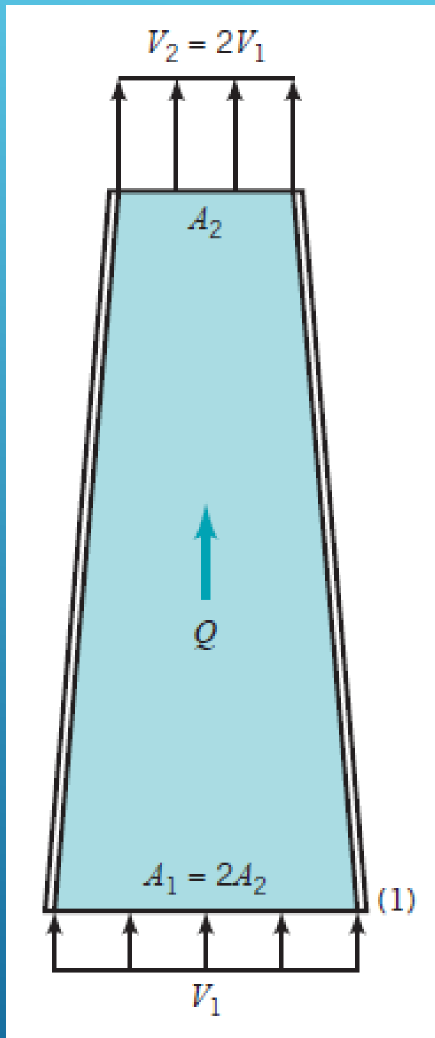
In many cases the fluid is physically constrained within a device so that its pressure cannot be prescribed a priori as was done for the free jet examples above. Such cases include nozzles and pipes of variable diameter for which the fluid velocity changes because the flow area is different from one section to another. For these situations it is necessary to use the concept of conservation of mass (the continuity equation) along with the Bernoulli equation. The derivation and use of this equation are discussed in detail in Chapters 4 and 5.

The *mass flowrate* from an outlet, $\dot{m}$ (slugs/s or kg/s), is given by $\dot{m} = \rho Q$, where Q (ft^3/s or m^3/s) is the ***volume flowrate.*** If the outlet area is A and the fluid flows across this area (normal to the area) with an average velocity V, then the volume of the fluid crossing this area in a time interval δt is $VA\,\delta t$, equal to that in a volume of length $V\,\delta t$ and cross-sectional area A (see Fig. 3.15*b*). Hence, the volume flowrate (volume per unit time) is $Q = VA$. Thus, $\dot{m} = \rho VA$. To conserve mass, the inflow rate must equal the outflow rate. If the inlet is designated as (1) and the outlet as (2), it follows that $\dot{m}_1 = \dot{m}_2$. Thus, conservation of mass requires

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

If the density remains constant, then $\rho_1 = \rho_2$, and the above becomes the ***continuity equation*** for incompressible flow

$$A_1 V_1 = A_2 V_2, \text{ or } Q_1 = Q_2 \qquad \textbf{(3.19)}$$



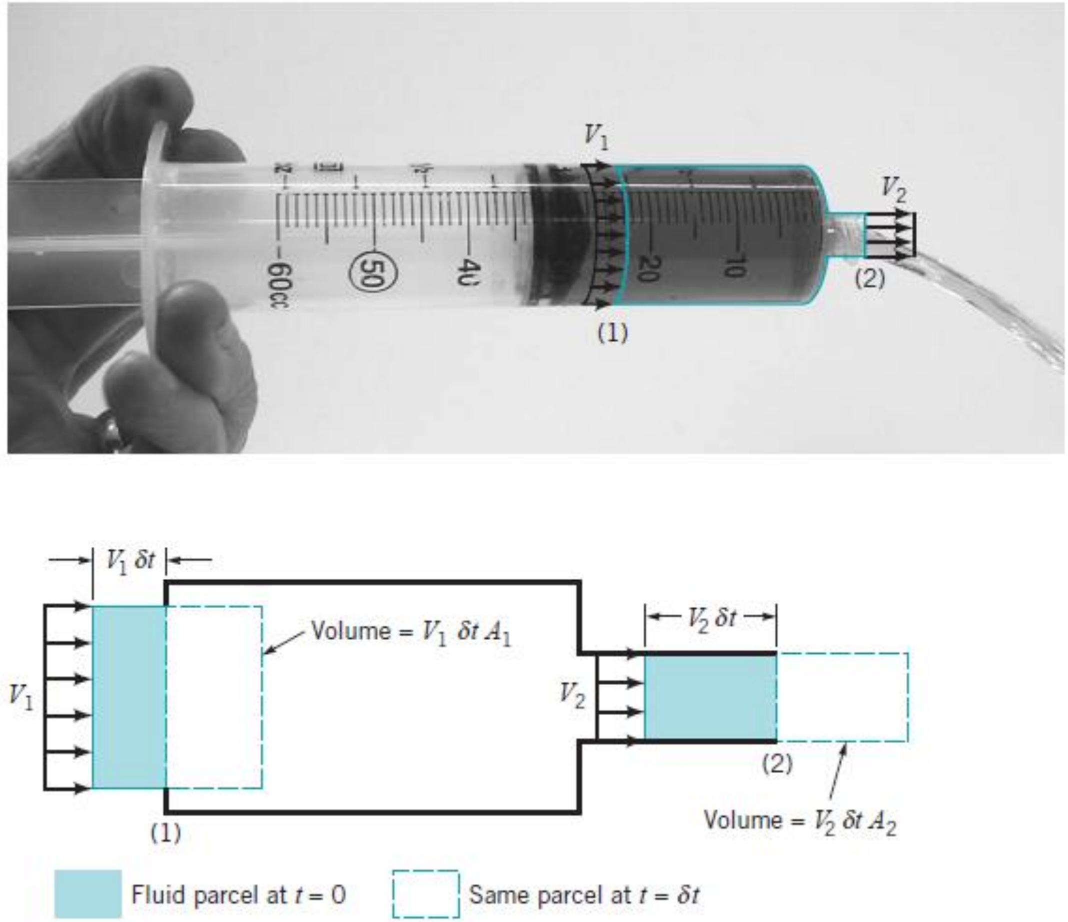
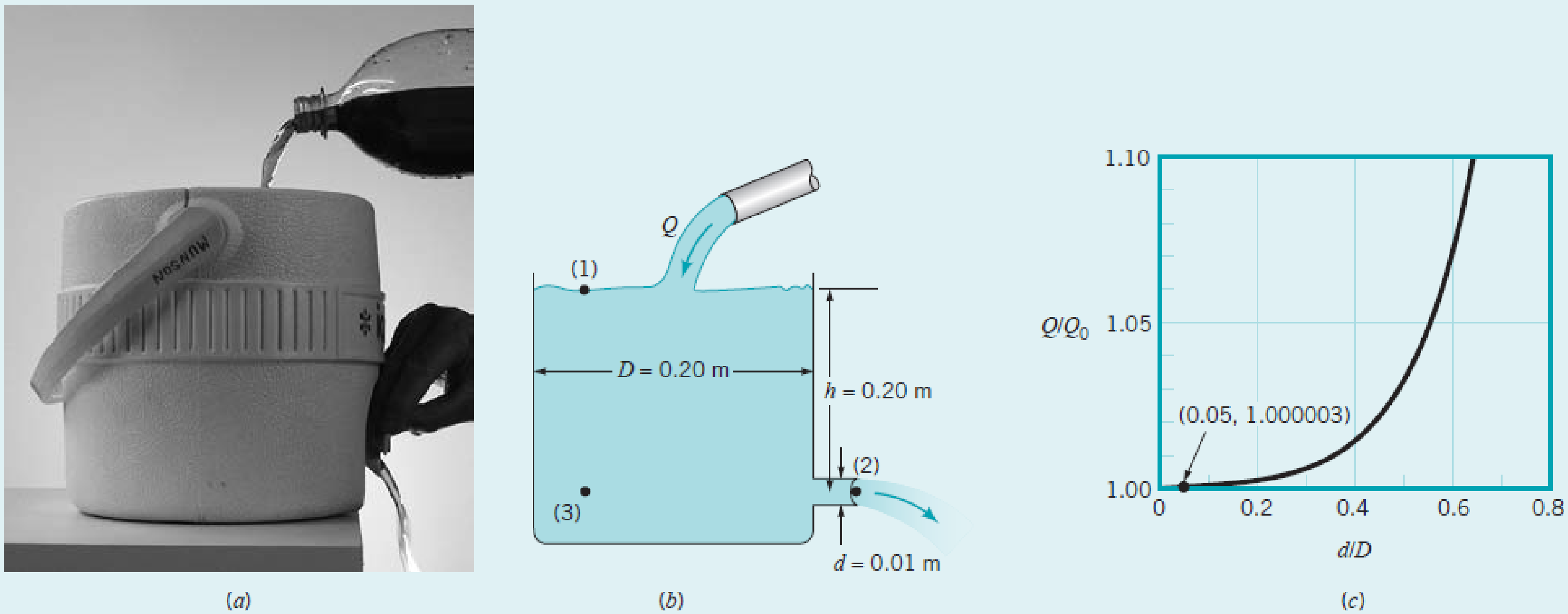


FIGURE 3.15 (*a*) Flow through a syringe. (*b*) Steady flow into and out of a volume.

EXAMPLE 3.7 Flow from a Tank—Gravity

GIVEN A stream of refreshing beverage of diameter $d = 0.01$ m flows steadily from the cooler of diameter $D = 0.20$ m as shown in Figs. E3.7*a* and *b*.

FIND Determine the flowrate, Q, from the bottle into the cooler if the depth of beverage in the cooler is to remain constant at $h = 0.20$ m



■ FIGURE E3.7

SOLUTION

For steady, inviscid, incompressible flow, the Bernoulli equation applied between points (1) and (2) is

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2 \quad \textbf{(1)}$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2 \quad \textbf{(2)}$$

Although the liquid level remains constant (h = constant), there is an average velocity, V_1, across section (1) because of the flow from the tank. From Eq. 3.19 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2 \quad \textbf{(3)}$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}} = \sqrt{\frac{2(9.81\ \text{m/s}^2)(0.20\ \text{m})}{1-(0.01\ \text{m}/0.20\ \text{m})^4}} = 1.98\ \text{m/s}$$

Thus,

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.01\ \text{m})^2(1.98\ \text{m/s})$$
$$= 1.56 \times 10^{-4}\ \text{m}^3/\text{s} \quad \textbf{(Ans)}$$

COMMENTS Note that this problem was solved using points (1) and (2) located at the free surface and the exit of the pipe, respectively. Although this was convenient (because most of the variables are known at those points), other points could be selected and the same result would be obtained. For example, consider points (1) and (3) as indicated in Fig. E3.7*b*. At (3), located sufficiently far from the tank exit, $V_3 = 0$ and $z_3 = z_2 = 0$. Also, $p_3 = \gamma h$ since the pressure is hydrostatic sufficiently far from the exit. Use of this information in the Bernoulli equation applied between (1) and (3) gives the exact same result as obtained using it between (1) and (2). The only difference is that the elevation head, $z_1 = h$, has been interchanged with the pressure head at (3), $p_3/\gamma = h$.

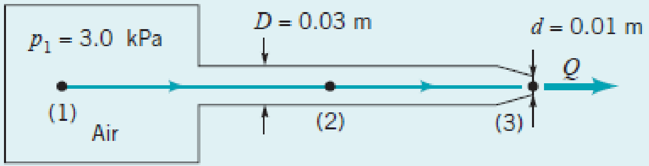
In this example we have not neglected the kinetic energy of the water in the tank ($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D \gg d$), Eq. 3 indicates that $V_1 \ll V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error associated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq 0$, denoted Q, to that assuming $V_1 = 0$, denoted Q_0. This ratio, written as

$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{D=\infty}} = \frac{\sqrt{2gh/[1-(d/D)^4]}}{\sqrt{2gh}} = \frac{1}{\sqrt{1-(d/D)^4}}$$

is plotted in Fig. E3.7*c*. With $0 < d/D < 0.4$ it follows that $1 < Q/Q_0 \lesssim 1.01$, and the error in assuming $V_1 = 0$ is less than 1%. For this example with $d/D = 0.01\ \text{m}/0.20\ \text{m} = 0.05$, it follows that $Q/Q_0 = 1.000003$. Thus, it is often reasonable to assume $V_1 = 0$.

EXAMPLE 3.8 Flow from a Tank—Pressure

GIVEN Air flows steadily from a tank, through a hose of diameter $D = 0.03$ m, and exits to the atmosphere from a nozzle of diameter $d = 0.01$ m as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure.



■ FIGURE E3.8

FIND Determine the flowrate and the pressure in the hose.

SOLUTION

If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet), this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2 \qquad \textbf{(1)}$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

$$\begin{aligned}\rho &= \frac{p_1}{RT_1} \\ &= [(3.0 + 101)\ \mathrm{kN/m^2}] \\ &\quad \times \frac{10^3\ \mathrm{N/kN}}{(286.9\ \mathrm{N \cdot m/kg \cdot K})(15 + 273)\mathrm{K}} \\ &= 1.26\ \mathrm{kg/m^3}\end{aligned}$$

Thus, we find that

$$V_3 = \sqrt{\frac{2(3.0 \times 10^3\ \mathrm{N/m^2})}{1.26\ \mathrm{kg/m^3}}} = 69.0\ \mathrm{m/s}$$

or

$$\begin{aligned}Q &= A_3 V_3 = \frac{\pi}{4} d^2 V_3 = \frac{\pi}{4}(0.01\ \mathrm{m})^2 (69.0\ \mathrm{m/s}) \\ &= 0.00542\ \mathrm{m^3/s}\end{aligned}$$

(Ans)

The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 3.19)

$$A_2V_2 = A_3V_3$$

Hence,

$$V_2 = A_3V_3/A_2 = \left(\frac{d}{D}\right)^2 V_3$$

$$= \left(\frac{0.01\text{ m}}{0.03\text{ m}}\right)^2 (69.0\text{ m/s}) = 7.67\text{ m/s}$$

and from Eq. 1

$$p_2 = 3.0 \times 10^3\text{ N/m}^2 - \tfrac{1}{2}(1.26\text{ kg/m}^3)(7.67\text{ m/s})^2$$

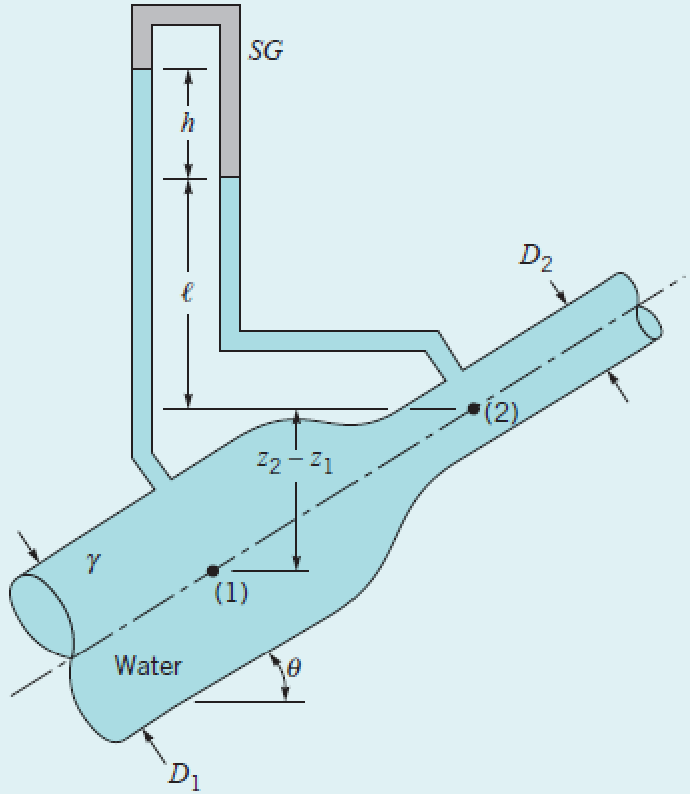
$$= (3000 - 37.1)\text{N/m}^2 = 2963\text{ N/m}^2 \qquad \textbf{(Ans)}$$

COMMENTS Note that the value of V_3 is determined strictly by the value of p_1 (and the assumptions involved in the Bernoulli equation), independent of the "shape" of the nozzle. The pressure head within the tank, $p_1/\gamma = (3.0\text{ kPa})/(9.81\text{ m/s}^2)(1.26\text{ kg/m}^3) =$ 243 m, is converted to the velocity head at the exit, $V_2^2/2g = (69.0\text{ m/s})^2/(2 \times 9.81\text{ m/s}^2) = 243$ m. Although we used gage pressure in the Bernoulli equation ($p_3 = 0$), we had to use absolute pressure in the perfect gas law when calculating the density.

EXAMPLE 3.9 Flow in a Variable Area Pipe

GIVEN Water flows through a pipe reducer as is shown in Fig. E3.9. The static pressures at (1) and (2) are measured by the inverted U-tube manometer containing oil of specific gravity, *SG*, less than one.

FIND Determine the manometer reading, *h*.



■ FIGURE E3.9

Solution

With the assumptions of steady, inviscid, incompressible flow, the Bernoulli equation can be written as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

The continuity equation (Eq. 3.19) provides a second relationship between V_1 and V_2 if we assume the velocity profiles are uniform at those two locations and the fluid incompressible:

$$Q = A_1 V_1 = A_2 V_2$$

By combining these two equations we obtain

$$p_1 - p_2 = \gamma(z_2 - z_1) + \tfrac{1}{2}\rho V_2^2 [1 - (A_2/A_1)^2] \qquad \textbf{(1)}$$

This pressure difference is measured by the manometer and can be determined by using the pressure–depth ideas developed in Chapter 2. Thus,

$$p_1 - \gamma(z_2 - z_1) - \gamma\ell - \gamma h + SG\,\gamma h + \gamma\ell = p_2$$

or

$$p_1 - p_2 = \gamma(z_2 - z_1) + (1 - SG)\gamma h \qquad \textbf{(2)}$$

As discussed in Chapter 2, this pressure difference is neither merely γh nor $\gamma(h + z_1 - z_2)$.

Equations 1 and 2 can be combined to give the desired result as follows:

$$(1 - SG)\gamma h = \frac{1}{2}\rho V_2^2 \left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

or since $V_2 = Q/A_2$

$$h = (Q/A_2)^2 \frac{1 - (A_2/A_1)^2}{2g(1 - SG)} \qquad \textbf{(Ans)}$$

If the differences in velocity are considerable, the differences in pressure can also be considerable. For flows of gases, this may introduce compressibility effects as discussed in Section 3.8 and Chapter 11. For flows of liquids, this may result in ***cavitation***, a potentially dangerous situation that results when the liquid pressure is reduced to the vapor pressure and the liquid "boils."

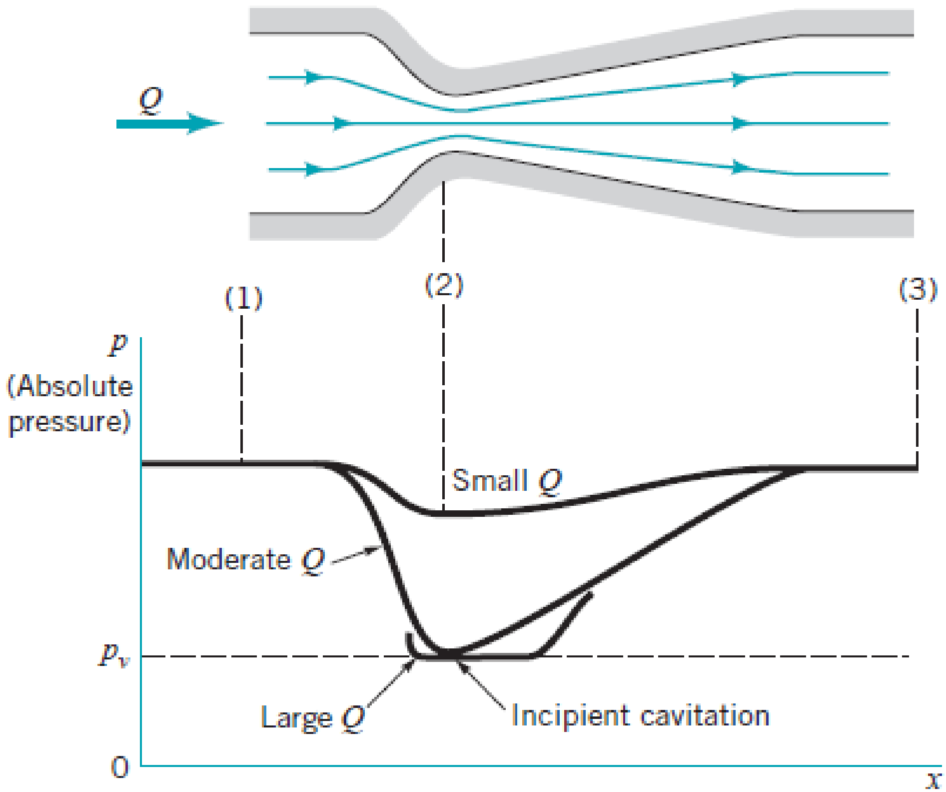


FIGURE 3.16 Pressure variation and cavitation in a variable area pipe.

EXAMPLE 3.10 Siphon and Cavitation

GIVEN A liquid can be siphoned from a container as shown in Fig. E3.10*a* provided the end of the tube, point (3), is below the free surface in the container, point (1), and the maximum elevation of the tube, point (2), is "not too great." Consider water at 60° F being siphoned from a large tank through a constant diameter hose as shown in Fig. E3.10*b*. The end of the siphon is 5 ft below the bottom of the tank, and the atmospheric pressure is 14.7 psia.

FIND Determine the maximum height of the hill, H, over which the water can be siphoned without cavitation occurring.

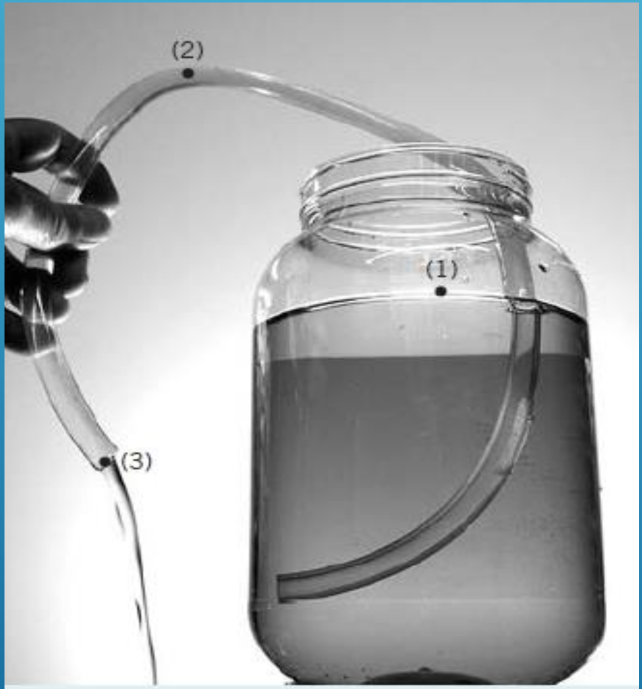


FIGURE E3.10*a*

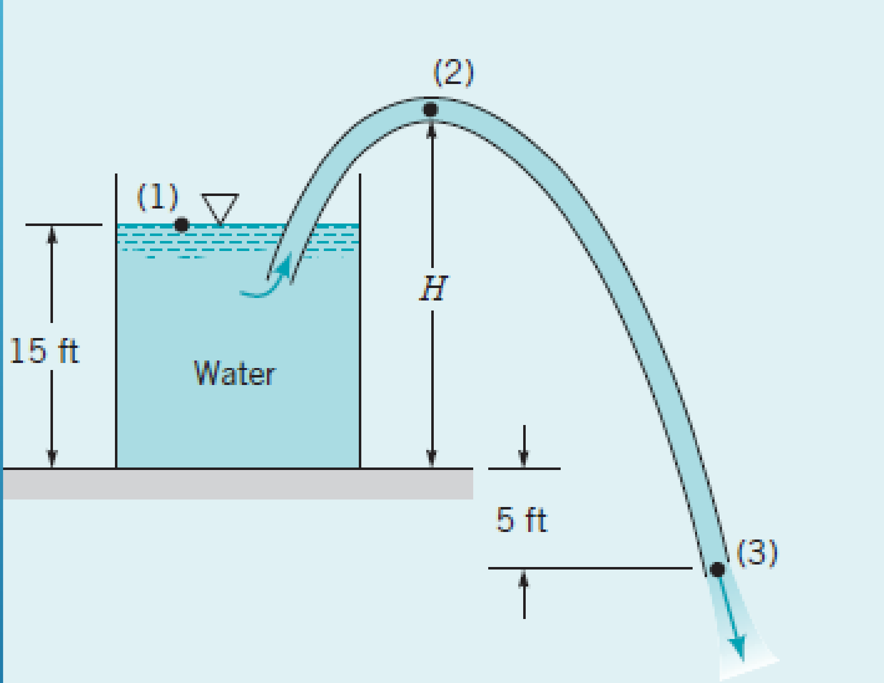


FIGURE E3.10*b*

SOLUTION

If the flow is steady, inviscid, and incompressible we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as follows:

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3 \qquad \textbf{(1)}$$

With the tank bottom as the datum, we have $z_1 = 15$ ft, $z_2 = H$, and $z_3 = -5$ ft. Also, $V_1 = 0$ (large tank), $p_1 = 0$ (open tank), $p_3 = 0$ (free jet), and from the continuity equation $A_2V_2 = A_3V_3$, or because the hose is constant diameter, $V_2 = V_3$. Thus, the speed of the fluid in the hose is determined from Eq. 1 to be

$$V_3 = \sqrt{2g(z_1 - z_3)} = \sqrt{2(32.2 \text{ ft/s}^2)[15 - (-5)] \text{ ft}}$$
$$= 35.9 \text{ ft/s} = V_2$$

Use of Eq. 1 between points (1) and (2) then gives the pressure p_2 at the top of the hill as

$$p_2 = p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 - \tfrac{1}{2}\rho V_2^2 - \gamma z_2$$
$$= \gamma(z_1 - z_2) - \tfrac{1}{2}\rho V_2^2 \qquad \textbf{(2)}$$

From Table B.1, the vapor pressure of water at 60 °F is 0.256 psia. Hence, for incipient cavitation the lowest pressure in the system will be $p = 0.256$ psia. Careful consideration of Eq. 2 and Fig. E3.10*b* will show that this lowest pressure will occur at the top of the hill. Since we have used gage pressure at point (1) ($p_1 = 0$), we must use gage pressure at point (2) also. Thus, $p_2 = 0.256 - 14.7 = -14.4$ psi and Eq. 2 gives

$$(-14.4 \text{ lb/in.}^2)(144 \text{ in.}^2/\text{ft}^2) = (62.4 \text{ lb/ft}^3)(15 - H)\text{ft} - \tfrac{1}{2}(1.94 \text{ slugs/ft}^3)(35.9 \text{ ft/s})^2$$

or

$$H = 28.2 \text{ ft} \qquad \textbf{(Ans)}$$

For larger values of H, vapor bubbles will form at point (2) and the siphon action may stop.

3.6.3 Flowrate Measurement

Many types of devices using principles involved in the Bernoulli equation have been developed to measure fluid velocities and flowrates.

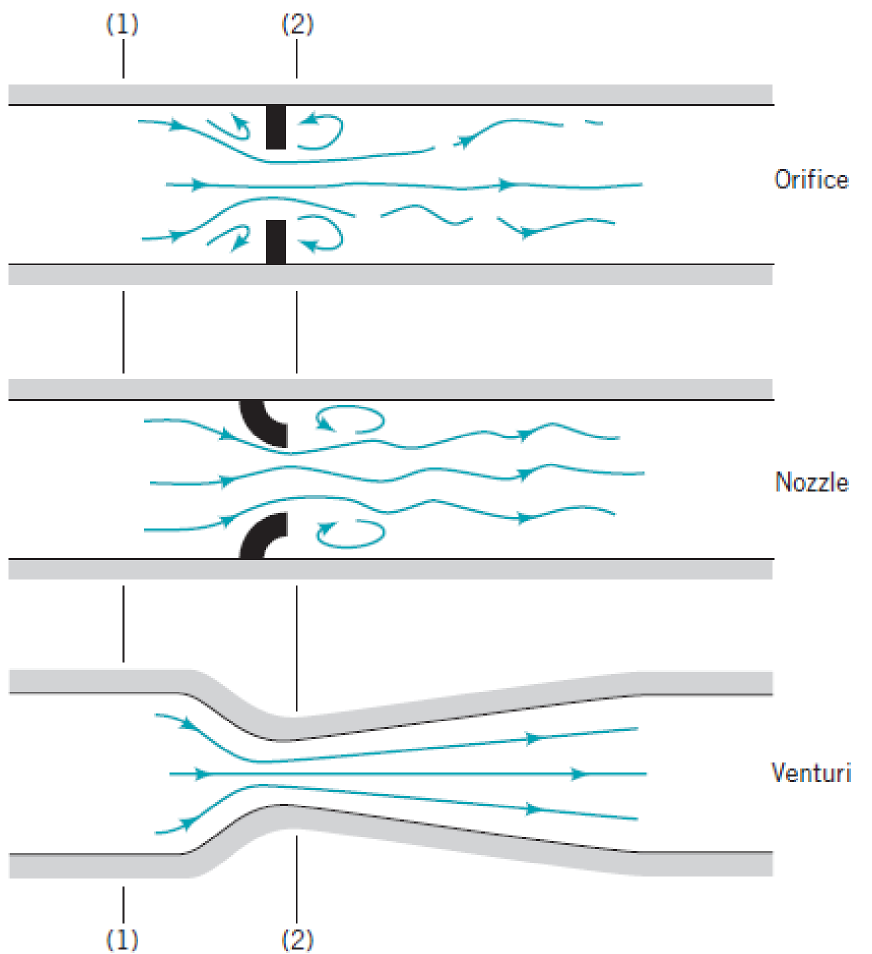


FIGURE 3.18 Typical devices for measuring flowrate in pipes.

An effective way to measure the flowrate through a pipe is to place some type of restriction within the pipe as shown in Fig. 3.18 and to measure the pressure difference between the low-velocity, high-pressure upstream section (1), and the high-velocity, low-pressure downstream section (2). Three commonly used types of flow meters are illustrated: the *orifice meter*, the *nozzle meter*, and the *Venturi meter*. The operation of each is based on the same physical principles—an increase in velocity causes a decrease in pressure. The difference between them is a matter of cost, accuracy, and how closely their actual operation obeys the idealized flow assumptions.

We assume the flow is horizontal ($z_1 = z_2$), steady, inviscid, and incompressible between points (1) and (2). The Bernoulli equation becomes

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_2^2$$

(The effect of nonhorizontal flow can be incorporated easily by including the change in elevation, $z_1 - z_2$, in the Bernoulli equation.)

If we assume the velocity profiles are uniform at sections (1) and (2), the continuity equation (Eq. 3.19) can be written as

$$Q = A_1V_1 = A_2V_2$$

where A_2 is the small ($A_2 < A_1$) flow area at section (2). Combination of these two equations results in the following theoretical flowrate

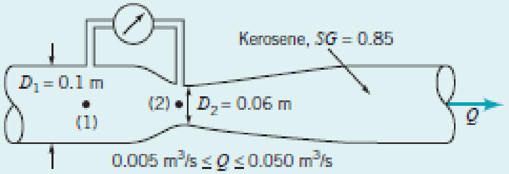
$$Q = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}} \quad \textbf{(3.20)}$$

Thus, as shown by the figure in the margin, for a given flow geometry (A_1 and A_2) the flowrate can be determined if the pressure difference, $p_1 - p_2$, is measured. The actual measured flowrate, Q_{actual}, will be smaller than this theoretical result because of various differences between the "real world" and the assumptions used in the derivation of Eq. 3.20. These differences (which are quite consistent and may be as small as 1 to 2% or as large as 40%, depending on the geometry used) can be accounted for by using an empirically obtained discharge coefficient as discussed in Section 8.6.1.

EXAMPLE 3.11 Venturi Meter

GIVEN Kerosene ($SG = 0.85$) flows through the Venturi meter shown in Fig. E3.11*a* with flowrates between 0.005 and 0.050 m^3/s.

FIND Determine the range in pressure difference, $p_1 - p_2$, needed to measure these flowrates.



■ FIGURE E3.11*a*

SOLUTION

If the flow is assumed to be steady, inviscid, and incompressible, the relationship between flowrate and pressure is given by Eq. 3.20. This can be rearranged to give

$$p_1 - p_2 = \frac{Q^2\rho[1 - (A_2/A_1)^2]}{2\,A_2^2}$$

With the density of the flowing fluid

$$\rho = SG\,\rho_{H_2O} = 0.85(1000\text{ kg/m}^3) = 850\text{ kg/m}^3$$

and the area ratio

$$A_2/A_1 = (D_2/D_1)^2 = (0.06\text{ m}/0.10\text{ m})^2 = 0.36$$

the pressure difference for the smallest flowrate is

$$p_1 - p_2 = (0.005\text{ m}^3\text{/s})^2(850\text{ kg/m}^3)\frac{(1 - 0.36^2)}{2\,[(\pi/4)(0.06\text{ m})^2]^2}$$

$$= 1160\text{ N/m}^2 = 1.16\text{ kPa}$$

Likewise, the pressure difference for the largest flowrate is

$$p_1 - p_2 = (0.05)^2(850)\frac{(1 - 0.36^2)}{2[(\pi/4)(0.06)^2]^2}$$

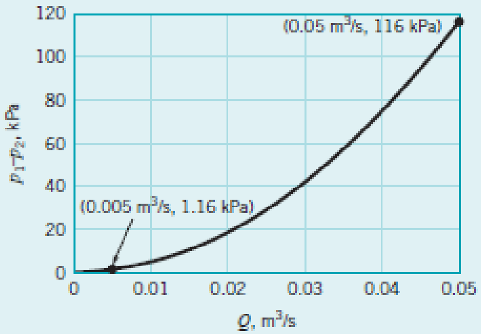
$$= 1.16 \times 10^5\text{ N/m}^2 = 116\text{ kPa}$$

Thus,

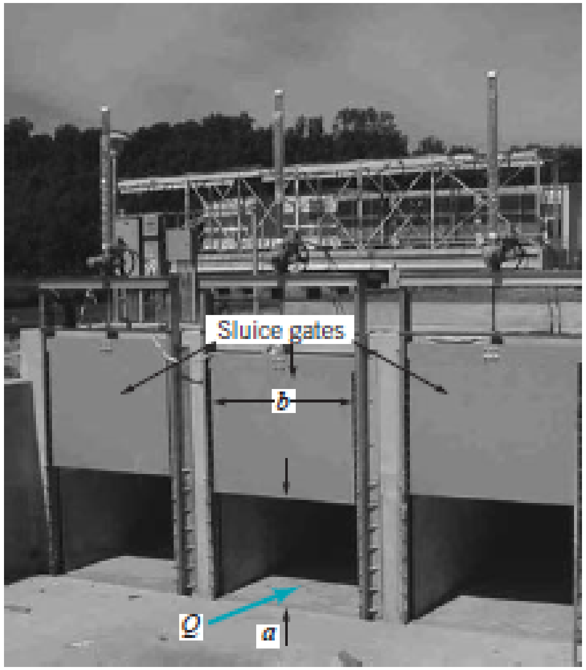
$$1.16\text{ kPa} \leq p_1 - p_2 \leq 116\text{ kPa} \qquad \text{(Ans)}$$

COMMENTS These values represent the pressure differences for inviscid, steady, incompressible conditions. The ideal results presented here are independent of the particular flow meter geometry—an orifice, nozzle, or Venturi meter (see Fig. 3.18).

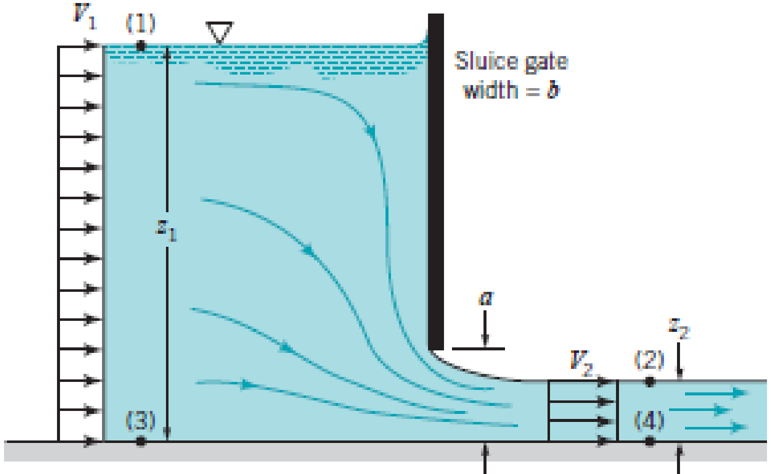
It is seen from Eq. 3.20 that the flowrate varies as the square root of the pressure difference. Hence, as indicated by the numerical results and shown in Fig. E3.11*b*, a 10-fold increase in flowrate requires a 100-fold increase in pressure difference. This nonlinear relationship can cause difficulties when measuring flowrates over a wide range of values. Such measurements would require pressure transducers with a wide range of operation. An alternative is to use two flow meters in parallel—one for the larger and one for the smaller flowrate ranges.



■ FIGURE E3.11*b*



(a)



(b)

FIGURE 3.19 Sluice gate geometry. (Photograph courtesy of Plasti-Fab, Inc.)

Thus, we apply the Bernoulli equation between points on the free surfaces at (1) and (2) to give

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

Also, if the gate is the same width as the channel so that $A_1 = bz_1$ and $A_2 = bz_2$, the continuity equation gives

$$Q = A_1V_1 = bV_1z_1 = A_2V_2 = bV_2z_2$$

With the fact that $p_1 = p_2 = 0$, these equations can be combined and rearranged to give the flowrate as

$$Q = z_2 b \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}} \qquad \textbf{(3.21)}$$

In the limit of $z_1 \gg z_2$ this result simply becomes

$$Q = z_2 b \sqrt{2gz_1}$$

This limiting result represents the fact that if the depth ratio, z_1/z_2, is large, the kinetic energy of the fluid upstream of the gate is negligible and the fluid velocity after it has fallen a distance $(z_1 - z_2) \approx z_1$ is approximately $V_2 = \sqrt{2gz_1}$.

The results of Eq. 3.21 could also be obtained by using the Bernoulli equation between points (3) and (4) and the fact that $p_3 = \gamma z_1$ and $p_4 = \gamma z_2$ since the streamlines at these sections are straight. In this formulation, rather than the potential energies at (1) and (2), we have the pressure contributions at (3) and (4).

The downstream depth, z_2, not the gate opening, a, was used to obtain the result of Eq. 3.21. As was discussed relative to flow from an orifice (Fig. 3.14), the fluid cannot turn a sharp 90° corner. A vena contracta results with a contraction coefficient, $C_c = z_2/a$, less than 1. Typically C_c is approximately 0.61 over the depth ratio range of $0 < a/z_1 < 0.2$. For larger values of a/z_1 the value of C_c increases rapidly.

EXAMPLE 3.12 Sluice Gate

GIVEN Water flows under the sluice gate shown in Fig. E3.12*a*.

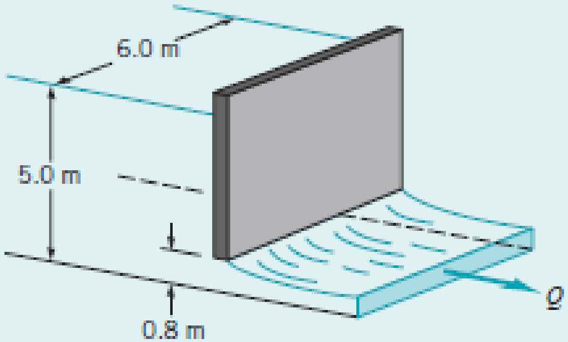
FIND Determine the approximate flowrate per unit width of the channel.

SOLUTION

Under the assumptions of steady, inviscid, incompressible flow, we can apply Eq. 3.21 to obtain Q/b, the flowrate per unit width, as

$$\frac{Q}{b} = z_2 \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}}$$

In this instance $z_1 = 5.0$ m and $a = 0.80$ m so the ratio $a/z_1 = 0.16 < 0.20$, and we can assume that the contraction coefficient is approximately $C_c = 0.61$. Thus, $z_2 = C_c a = 0.61$ (0.80 m) = 0.488 m and we obtain the flowrate



■ FIGURE E3.12*a*

$$\frac{Q}{b} = (0.488\text{ m})\sqrt{\frac{2(9.81\text{ m/s}^2)(5.0\text{ m} - 0.488\text{ m})}{1 - (0.488\text{ m}/5.0\text{ m})^2}}$$

$$= 4.61\text{ m}^2\text{/s} \qquad \textbf{(Ans)}$$

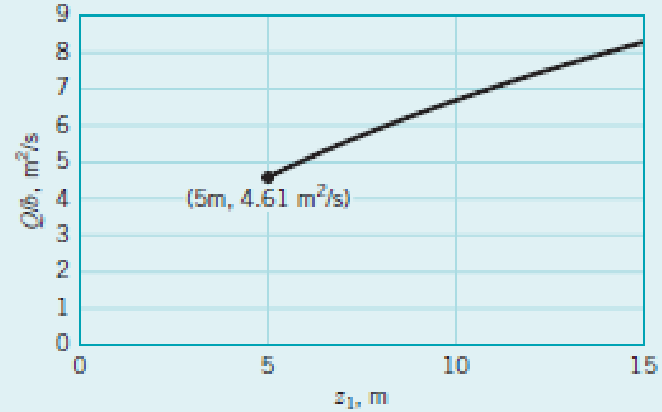
COMMENT If we consider $z_1 \gg z_2$ and neglect the kinetic energy of the upstream fluid, we would have

$$\frac{Q}{b} = z_2\sqrt{2gz_1} = 0.488\text{ m}\sqrt{2(9.81\text{ m/s}^2)(5.0\text{ m})}$$

$$= 4.83\text{ m}^2\text{/s}$$

In this case the difference in Q with or without including V_1 is not too significant because the depth ratio is fairly large ($z_1/z_2 = 5.0/0.488 = 10.2$). Thus, it is often reasonable to neglect the kinetic energy upstream from the gate compared to that downstream of it.

By repeating the calculations for various flow depths, z_1, the results shown in Fig. E3.12*b* are obtained. Note that the flowrate is not directly proportional to the flow depth. Thus, for example, if during flood conditions the upstream depth doubled from $z_1 = 5$ m to $z_1 = 10$ m, the flowrate per unit width of the channel would not double, but would increase only from 4.61 m²/s to 6.67 m²/s.



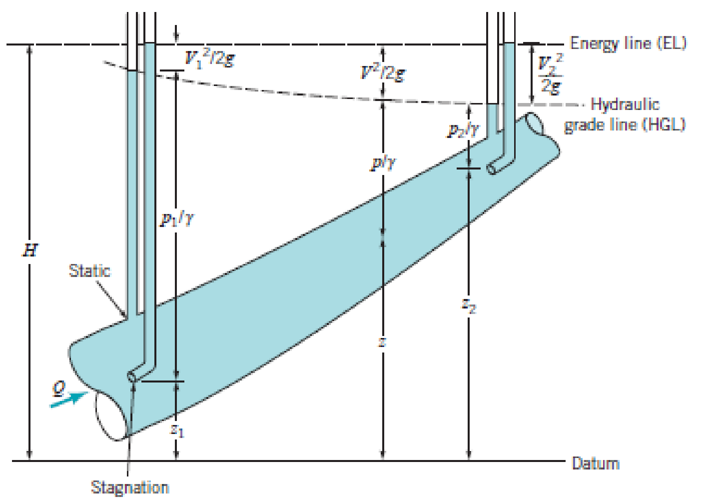
■ FIGURE E3.12*b*

3.7 The Energy Line and the Hydraulic Grade Line

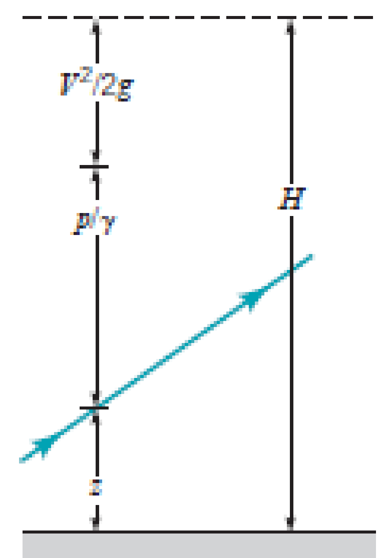
As was discussed in Section 3.4, the Bernoulli equation is actually an energy equation representing the partitioning of energy for an inviscid, incompressible, steady flow. The sum of the various energies of the fluid remains constant as the fluid flows from one section to another. A useful interpretation of the Bernoulli equation can be obtained through the use of the concepts of the *hydraulic grade line* (HGL) and the *energy line* (EL). These ideas represent a geometrical interpretation of a flow and can often be effectively used to better grasp the fundamental processes involved.

For steady, inviscid, incompressible flow the total energy remains constant along a streamline. The concept of "head" was introduced by dividing each term in Eq. 3.7 by the specific weight, $\gamma = \rho g$, to give the Bernoulli equation in the following form

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline} = H \quad (3.22)$$



■ FIGURE 3.21 Representation of the energy line and the hydraulic grade line.



Each of the terms in this equation has the units of length (feet or meters) and represents a certain type of head. The Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline. This constant is called the *total head*, H.

The energy line is a line that represents the total head available to the fluid. As shown in Fig. 3.21, the elevation of the energy line can be obtained by measuring the stagnation pressure with a Pitot tube. (A Pitot tube is the portion of a Pitot-static tube that measures the stagnation pressure. See Section 3.5.) The stagnation point at the end of the Pitot tube provides a measurement of the total head (or energy) of the flow. The static pressure tap connected to the piezometer tube shown, on the other hand, measures the sum of the pressure head and the elevation head, $p/\gamma + z$. This sum is often called the *piezometric head*. The static pressure tap does not measure the velocity head.

According to Eq. 3.22, the total head remains constant along the streamline (provided the assumptions of the Bernoulli equation are valid). Thus, a Pitot tube at any other location in the flow will measure the same total head, as is shown in the figure. The elevation head, velocity head, and pressure head may vary along the streamline, however.

The energy line and hydraulic grade line for flow from a large tank are shown in Fig. 3.22. If the flow is steady, incompressible, and inviscid, the energy line is horizontal and at the elevation of the liquid in the tank (since the fluid velocity in the tank and the pressure on the surface

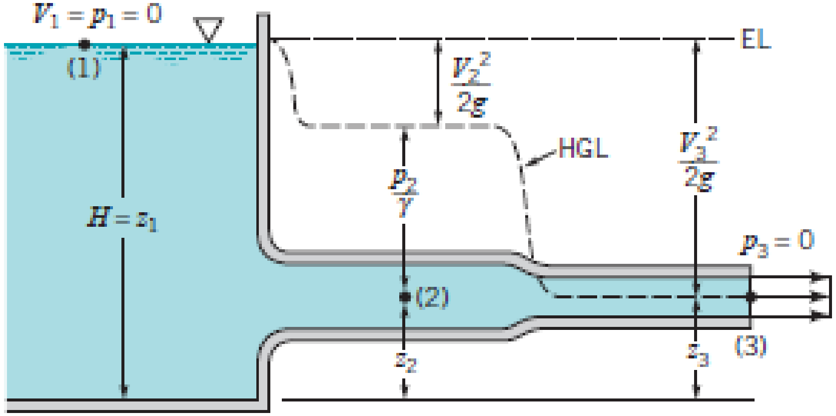


FIGURE 3.22 The energy line and hydraulic grade line for flow from a tank.

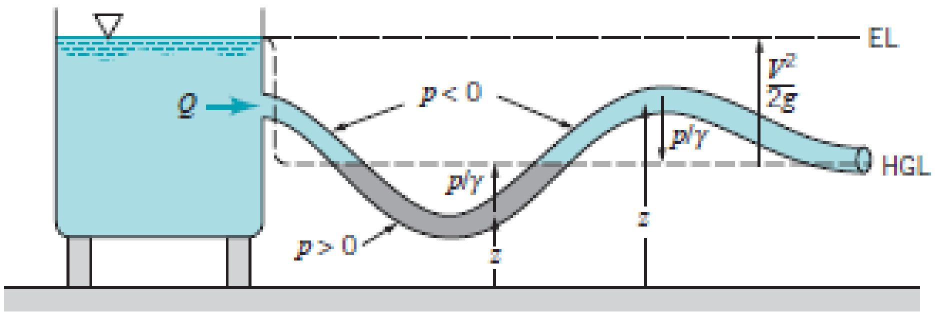


FIGURE 3.23 Use of the energy line and the hydraulic grade line.

are zero). The hydraulic grade line lies a distance of one velocity head, $V^2/2g$, below the energy line. Thus, a change in fluid velocity due to a change in the pipe diameter results in a change in the elevation of the hydraulic grade line. At the pipe outlet the pressure head is zero (gage) so the pipe elevation and the hydraulic grade line coincide.

3.8 Restrictions on Use of the Bernoulli Equation

Proper use of the Bernoulli equation requires close attention to the assumptions used in its derivation. In this section we review some of these assumptions and consider the consequences of incorrect use of the equation.

3.8.1 Compressibility Effects

One of the main assumptions is that the fluid is incompressible. Although this is reasonable for most liquid flows, it can, in certain instances, introduce considerable errors for gases.

In the previous section, we saw that the stagnation pressure, p_{stag}, is greater than the static pressure, p_{static}, by an amount $\Delta p = p_{stag} - p_{static} = \rho V^2/2$, provided that the density remains constant. If this dynamic pressure is not too large compared with the static pressure, the density change between two points is not very large and the flow can be considered incompressible. However, since the dynamic pressure varies as V^2, the error associated with the assumption that a fluid is incompressible increases with the square of the velocity of the fluid, as indicated by the figure in the margin. To account for compressibility effects we must return to Eq. 3.6 and properly integrate the term $\int dp/\rho$ when ρ is not constant.

A simple, although specialized, case of compressible flow occurs when the temperature of a perfect gas remains constant along the streamline—isothermal flow. Thus, we consider $p = \rho RT$, where T is constant. (In general, p, ρ, and T will vary.) For steady, inviscid, isothermal flow, Eq. 3.6 becomes

$$RT \int \frac{dp}{p} + \frac{1}{2}V^2 + gz = \text{constant}$$

where we have used $\rho = p/RT$. The pressure term is easily integrated and the constant of integration evaluated if z_1, p_1, and V_1 are known at some location on the streamline. The result is

$$\frac{V_1^2}{2g} + z_1 + \frac{RT}{g} \ln\left(\frac{p_1}{p_2}\right) = \frac{V_2^2}{2g} + z_2 \qquad (3.23)$$

Thus, the final form of Eq. 3.6 for compressible, isentropic, steady flow of a perfect gas is

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 \qquad \textbf{(3.24)}$$

We consider the stagnation point flow of Section 3.5 to illustrate the difference between the incompressible and compressible results. As is shown in Chapter 11, Eq. 3.24 can be written in dimensionless form as

$$\frac{p_2 - p_1}{p_1} = \left[\left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)^{k/k-1} - 1\right] \qquad \text{(compressible)} \qquad \textbf{(3.25)}$$

A comparison between this compressible result and the incompressible result is perhaps most easily seen if we write the incompressible flow result in terms of the pressure ratio and the Mach number. Thus, we divide each term in the Bernoulli equation, $\rho V_1^2/2 + p_1 = p_2$, by p_1 and use the perfect gas law, $p_1 = \rho RT_1$, to obtain

$$\frac{p_2 - p_1}{p_1} = \frac{V_1^2}{2RT_1}$$

Since $\mathrm{Ma}_1 = V_1/\sqrt{kRT_1}$ this can be written as

$$\frac{p_2 - p_1}{p_1} = \frac{k\mathrm{Ma}_1^2}{2} \qquad \text{(incompressible)} \qquad \textbf{(3.26)}$$

EXAMPLE 3.15 Compressible Flow—Mach Number

GIVEN The jet shown in Fig. E3.15 flies at Mach 0.82 at an altitude of 10 km in a standard atmosphere.

FIND Determine the stagnation pressure on the leading edge of its wing if the flow is incompressible; and if the flow is compressible isentropic.

FIGURE E3.15 (Photograph courtesy of Pure stock/superstock.)

SOLUTION

From Tables 1.8 and C.2 we find that $p_1 = 26.5$ kPa (abs), $T_1 = -49.9\ °C$, $\rho = 0.414\ kg/m^3$, and $k = 1.4$. Thus, if we assume incompressible flow, Eq. 3.26 gives

$$\frac{p_2 - p_1}{p_1} = \frac{k\mathrm{Ma}_1^2}{2} = 1.4\frac{(0.82)^2}{2} = 0.471$$

or

$$p_2 - p_1 = 0.471(26.5\ \mathrm{kPa}) = 12.5\ \mathrm{kPa} \qquad \text{(Ans)}$$

On the other hand, if we assume isentropic flow, Eq. 3.25 gives

$$\frac{p_2 - p_1}{p_1} = \left\{\left[1 + \frac{(1.4-1)}{2}(0.82)^2\right]^{1.4/(1.4-1)} - 1\right\} = 0.555$$

or

$$p_2 - p_1 = 0.555(26.5\ \mathrm{kPa}) = 14.7\ \mathrm{kPa} \qquad \text{(Ans)}$$

COMMENT We see that at Mach 0.82 compressibility effects are of importance. The pressure (and, to a first approximation, the lift and drag on the airplane; see Chapter 9) is approximately $14.7/12.5 = 1.18$ times greater according to the compressible flow calculations. This may be very significant. As discussed in Chapter 11, for Mach numbers greater than 1 (supersonic flow) the differences between incompressible and compressible results are often not only quantitative but also qualitative.

Note that if the airplane were flying at Mach 0.30 (rather than 0.82) the corresponding values would be $p_2 - p_1 = 1.670$ kPa for incompressible flow and $p_2 - p_1 = 1.707$ kPa for compressible flow. The difference between these two results is about 2%.

3.8.2 Unsteady Effects

Another restriction of the Bernoulli equation (Eq. 3.7) is the assumption that the flow is steady. For such flows, on a given streamline the velocity is a function of only s, the location along the streamline. That is, along a streamline $V = V(s)$. For unsteady flows the velocity is also a function of time, so that along a streamline $V = V(s, t)$. Thus when taking the time derivative of the velocity to obtain the streamwise acceleration, we obtain $a_s = \partial V/\partial t + V\,\partial V/\partial s$ rather than just $a_s = V\,\partial V/\partial s$ as is true for steady flow. For steady flows the acceleration is due to the change in velocity resulting from a change in position of the particle (the $V\,\partial V/\partial s$ term), whereas for unsteady flow there is an additional contribution to the acceleration resulting from a change in velocity with time at a fixed location (the $\partial V/\partial t$ term). These effects are discussed in detail in Chapter 4. The net effect is that the inclusion of the unsteady term, $\partial V/\partial t$, does not allow the equation of motion to be easily integrated (as was done to obtain the Bernoulli equation) unless additional assumptions are made.

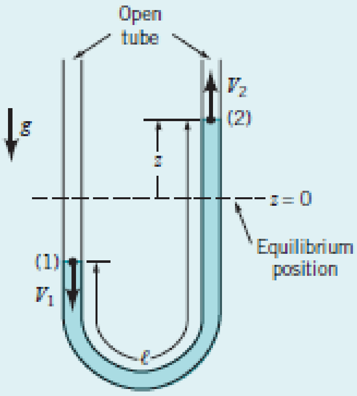
For incompressible flow this can be easily integrated between points (1) and (2) to give

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = \rho \int_{s_1}^{s_2} \frac{\partial V}{\partial t} ds + p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \qquad \text{(along a streamline)} \qquad (3.27)$$

EXAMPLE 3.16 Unsteady Flow—U-Tube

GIVEN An incompressible, inviscid liquid is placed in a vertical, constant diameter U-tube as indicated in Fig. E3.16. When released from the nonequilibrium position shown, the liquid column will oscillate at a specific frequency.

FIND Determine this frequency.



■ FIGURE E3.16

SOLUTION

The frequency of oscillation can be calculated by use of Eq. 3.27 as follows. Let points (1) and (2) be at the air–water interfaces of the two columns of the tube and $z = 0$ correspond to the equilibrium position of these interfaces. Hence, $p_1 = p_2 = 0$ and if $z_2 = z$, then $z_1 = -z$. In general, z is a function of time, $z = z(t)$. For a constant diameter tube, at any instant in time the fluid speed is constant throughout the tube, $V_1 = V_2 = V$, and the integral representing the unsteady effect in Eq. 3.27 can be written as

$$\int_{s_1}^{s_2} \frac{\partial V}{\partial t}\, ds = \frac{dV}{dt}\int_{s_1}^{s_2} ds = \ell\,\frac{dV}{dt}$$

where ℓ is the total length of the liquid column as shown in the figure. Thus, Eq. 3.27 can be written as

$$\gamma(-z) = \rho\ell\,\frac{dV}{dt} + \gamma z$$

Since $V = dz/dt$ and $\gamma = \rho g$, this can be written as the second-order differential equation describing simple harmonic motion

$$\frac{d^2 z}{dt^2} + \frac{2g}{\ell}z = 0$$

which has the solution $z(t) = C_1 \sin(\sqrt{2g/\ell}\; t) + C_2 \cos(\sqrt{2g/\ell}\; t)$. The values of the constants C_1 and C_2 depend on the initial state (velocity and position) of the liquid at $t = 0$. Thus, the liquid oscillates in the tube with a frequency

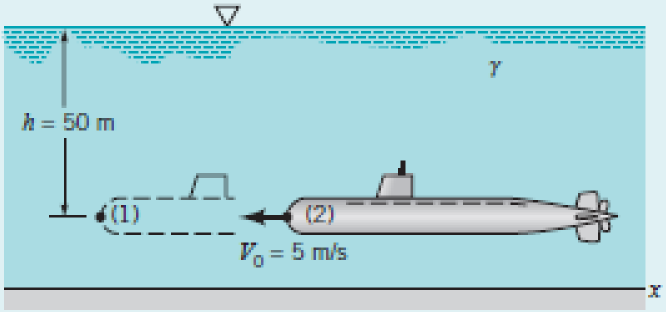
$$\omega = \sqrt{2g/\ell} \qquad \text{(Ans)}$$

COMMENT This frequency depends on the length of the column and the acceleration of gravity (in a manner very similar to the oscillation of a pendulum). The period of this oscillation (the time required to complete an oscillation) is $t_0 = 2\pi\sqrt{\ell/2g}$.

EXAMPLE 3.17 Unsteady or Steady Flow

GIVEN A submarine moves through seawater ($SG = 1.03$) at a depth of 50 m with velocity $V_0 = 5.0$ m/s as shown in Fig. E3.17.

FIND Determine the pressure at the stagnation point (2).



■ FIGURE E3.17

SOLUTION

In a coordinate system fixed to the ground, the flow is unsteady. For example, the water velocity at (1) is zero with the submarine in its initial position, but at the instant when the nose, (2), reaches point (1) the velocity there becomes $\mathbf{V}_1 = -V_0\hat{\mathbf{i}}$. Thus, $\partial \mathbf{V}_1/\partial t \neq 0$ and the flow is unsteady. Application of the steady Bernoulli equation between (1) and (2) would give the incorrect result that "$p_1 = p_2 + \rho V_0^2/2$." According to this result the static pressure is greater than the stagnation pressure—an incorrect use of the Bernoulli equation.

We can either use an unsteady analysis for the flow (which is outside the scope of this text) or redefine the coordinate system so that it is fixed on the submarine, giving steady flow with respect to this system. The correct method would be

$$p_2 = \frac{\rho V_1^2}{2} + \gamma h = [(1.03)(1000)\text{ kg/m}^3]\,(5.0\text{ m/s})^2/2 + (9.80 \times 10^3\text{ N/m}^3)(1.03)(50\text{ m})$$

$$= (12{,}900 + 505{,}000)\text{ N/m}^2$$

$$= 518\text{ kPa} \qquad \text{(Ans)}$$

similar to that discussed in Example 3.2.

COMMENT If the submarine were accelerating, $\partial V_0/\partial t \neq 0$, the flow would be unsteady in either of the above coordinate systems and we would be forced to use an unsteady form of the Bernoulli equation.

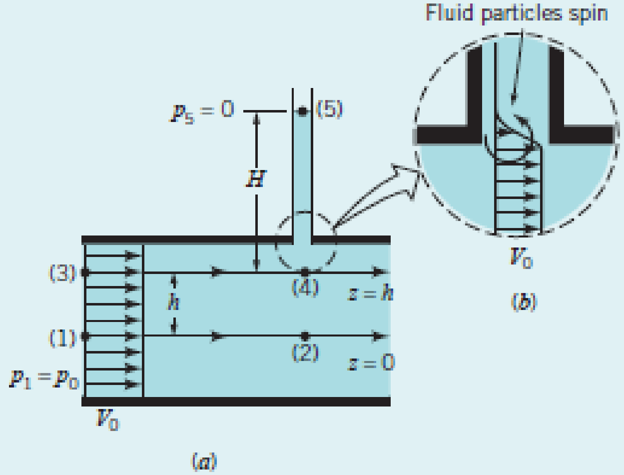
3.8.3 Rotational Effects

Another of the restrictions of the Bernoulli equation is that it is applicable along the streamline. Application of the Bernoulli equation across streamlines (i.e., from a point on one streamline to a point on another streamline) can lead to considerable errors, depending on the particular flow conditions involved. In general, the Bernoulli constant varies from streamline to streamline. However, under certain restrictions this constant is the same throughout the entire flow field.

EXAMPLE 3.18 Use of Bernoulli Equation across Streamlines

GIVEN Consider the uniform flow in the channel shown in Fig. E3.18*a*. The liquid in the vertical piezometer tube is stationary.

FIND Discuss the use of the Bernoulli equation between points (1) and (2), points (3) and (4), and points (4) and (5).



■ FIGURE E3.18

SOLUTION

If the flow is steady, inviscid, and incompressible, Eq. 3.7 written between points (1) and (2) gives

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= \text{constant} = C_{12}$$

Since $V_1 = V_2 = V_0$ and $z_1 = z_2 = 0$, it follows that $p_1 = p_2 = p_0$ and the Bernoulli constant for this streamline, C_{12}, is given by

$$C_{12} = \frac{1}{2}\rho V_0^2 + p_0$$

Along the streamline from (3) to (4) we note that $V_3 = V_4 = V_0$ and $z_3 = z_4 = h$. As was shown in Example 3.5, application of $\mathbf{F} = m\mathbf{a}$ across the streamline (Eq. 3.12) gives $p_3 = p_1 - \gamma h$ because the streamlines are straight and horizontal. The above facts combined with the Bernoulli equation applied between (3) and (4) show that $p_3 = p_4$ and that the Bernoulli constant along this streamline is the same as that along the streamline between (1) and (2). That is, $C_{34} = C_{12}$, or

$$p_3 + \frac{1}{2}\rho V_3^2 + \gamma z_3 = p_4 + \frac{1}{2}\rho V_4^2 + \gamma z_4 = C_{34} = C_{12}$$

Similar reasoning shows that the Bernoulli constant is the same for any streamline in Fig. E3.18. Hence,

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant throughout the flow}$$

Again from Example 3.5 we recall that

$$p_4 = p_5 + \gamma H = \gamma H$$

If we apply the Bernoulli equation across streamlines from (4) to (5), we obtain the incorrect result "$H = p_4/\gamma + V_4^2/2g$." The correct result is $H = p_4/\gamma$.

From the above we see that we can apply the Bernoulli equation across streamlines (1)–(2) and (3)–(4) (i.e., $C_{12} = C_{34}$) but not across streamlines from (4) to (5). The reason for this is that while the flow in the channel is "irrotational," it is "rotational" between the flowing fluid in the channel and the stationary fluid in the piezometer tube. Because of the uniform velocity profile across the channel, it is seen that the fluid particles do not rotate or "spin" as they move. The flow is "irrotational." However, as seen in Fig. E3.18*b*, there is a very thin shear layer between (4) and (5) in which adjacent fluid particles interact and rotate or "spin." This produces a "rotational" flow. A more complete analysis would show that the Bernoulli equation cannot be applied across streamlines if the flow is "rotational" (see Chapter 6).

Some of the important equations in this chapter are:

Streamwise and normal acceleration	$a_s = V\frac{\partial V}{\partial s}, \qquad a_n = \frac{V^2}{\mathfrak{R}}$	(3.1)
Force balance along a streamline for steady inviscid flow	$\int \frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C \qquad \text{(along a streamline)}$	(3.6)
The Bernoulli equation	$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline}$	(3.7)
Pressure gradient normal to streamline for inviscid flow in absence of gravity	$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathfrak{R}}$	(3.10b)
Force balance normal to a streamline for steady, inviscid, incompressible flow	$p + \rho \int \frac{V^2}{\mathfrak{R}}\, dn + \gamma z = \text{constant across the streamline}$	(3.12)
Velocity measurement for a Pitot-static tube	$V = \sqrt{2\,(p_3 - p_4)/\rho}$	(3.16)
Free jet	$V = \sqrt{2\frac{\gamma h}{\rho}} = \sqrt{2gh}$	(3.18)
Continuity equation	$A_1V_1 = A_2V_2, \text{ or } Q_1 = Q_2$	(3.19)
Flow meter equation	$Q = A_2\sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}}$	(3.20)
Sluice gate equation	$Q = z_2 b\sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}}$	(3.21)
Total head	$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline} = H$	(3.22)