MENG353 - FLUID MECHANICS

SOURCE: FUNDAMENTALS OF FLUIDMECHANICS
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CHAPTER 7: DIMENSIONAL ANALYSIS SMILITUDE, AND MODELING

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Learning Objectives

After completing this chapter, you should be able to:

- apply the Buckingham pi theorem.
- develop a set of dimensionless variables for a given flow situation.
- discuss the use of dimensionless variables in data analysis.
- apply the concepts of modeling and similitude to develop prediction equations.

Although many practical engineering problems involving fluid mechanics can be solved by using the equations and analytical procedures described in the preceding chapters, there remain a large number of problems that rely on experimentally obtained data for their solution. In fact, it is probably fair to say that very few problems involving real fluids can be solved by analysis alone. The solution to many problems is achieved through the use of a combination of theoretical and numerical analysis and experimental data.

An obvious goal of any experiment is to make the results as widely applicable as possible. To achieve this end, the concept of ***similitude*** is often used so that measurements made on one system (for example, in the laboratory) can be used to describe the behavior of other similar systems (outside the laboratory).

7.1 Dimensional Analysis

It is important to develop a meaningful and systematic way to perform an experiment.

The basis for this simplification lies in a consideration of the dimensions of the variables involved. As was discussed in Chapter 1, a qualitative description of physical quantities can be given in terms of ***basic dimensions*** such as mass, M, length, L, and time, T.

This type of analysis is called *dimensional analysis*, and the basis for its application to a wide variety of problems is found in the *Buckingham pi theorem* described in the following section.

7.2 Buckingham Pi Theorem

If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k - r$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

The dimensionless products are frequently referred to as "*pi terms*," and the theorem is called the *Buckingham pi theorem*.[2] Edgar Buckingham used the symbol Π to represent a dimensionless product, and this notation is commonly used. Although the pi theorem is a simple one, its proof is not so simple and we will not include it here.

The pi theorem is based on the idea of dimensional homogeneity which was introduced in Chapter 1. Essentially we assume that for any physically meaningful equation involving k variables, such as

$$u_1 = f(u_2, u_3, \ldots, u_k)$$

the dimensions of the variable on the left side of the equal sign must be equal to the dimensions of any term that stands by itself on the right side of the equal sign. It then follows that we can rearrange the equation into a set of dimensionless products (pi terms) so that

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

where $\phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$ is a function of Π_2 through Π_{k-r}.

The required number of pi terms is fewer than the number of original variables by r, where r is determined by the minimum number of reference dimensions required to describe the original list of variables. Usually the reference dimensions required to describe the variables will be the basic dimensions M, L, and T or F, L, and T. However, in some instances perhaps only two dimensions, such as L and T, are required, or maybe just one, such as L.

7.3 Determination of Pi Terms

The method we will describe in detail in this section is called the ***method of repeating variables***

Step 1 List all the variables that are involved in the problem.

Step 2 Express each of the variables in terms of basic dimensions.

Step 3 Determine the required number of pi terms.

Step 4 Select a number of repeating variables, where the number required is equal to the number of reference dimensions (usually the same as the number of basic dimensions).

Step 5 Form a pi term by multiplying one of the nonrepeating variables by the product of repeating variables each raised to an exponent that will make the combination dimensionless.

Step 6 Repeat Step 5 for each of the remaining nonrepeating variables.

Step 7 Check all the resulting pi terms to make sure they are dimensionless and independent.

Step 8 Express the final form as a relationship among the pi terms and think about what it means.

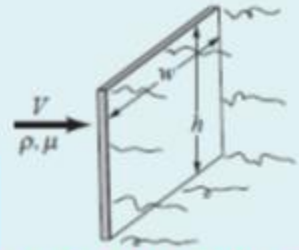
By using dimensional analysis, the original problem is simplified and defined with pi terms.

Special attention should be given to the selection of repeating variables as detailed in Step 4.

EXAMPLE 7.1 Method of Repeating Variables

GIVEN A thin rectangular plate having a width w and a height h is located so that it is normal to a moving stream of fluid as shown in Fig. E7.1. Assume the drag, $\mathcal{D}$, that the fluid exerts on the plate is a function of w and h, the fluid viscosity and density, μ and ρ, respectively, and the velocity V of the fluid approaching the plate.

FIND Determine a suitable set of pi terms to study this problem experimentally.



■ FIGURE E7.1

V7.2 Flow past a flat plate

SOLUTION

From the statement of the problem we can write

$$\mathcal{D} = f(w, h, \mu, \rho, V)$$

where this equation expresses the general functional relationship between the drag and the several variables that will affect it. The dimensions of the variables (using the *MLT* system) are

$$\mathcal{D} \doteq MLT^{-2}$$

$$w \doteq L$$

$$h \doteq L$$

$$\mu \doteq ML^{-1}T^{-1}$$

$$\rho \doteq ML^{-3}$$

$$V \doteq LT^{-1}$$

$$k - r = 6 - 3$$

$$\Pi_1 = \mathcal{D} w^a V^b \rho^c$$

and in terms of dimensions

$$(MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

Thus, for Π_1 to be dimensionless it follows that

$$1 + c = 0 \qquad (\text{for } M)$$

$$1 + a + b - 3c = 0 \qquad (\text{for } L)$$

$$-2 - b = 0 \qquad (\text{for } T)$$

and, therefore, $a = -2$, $b = -2$, and $c = -1$. The pi term then becomes

$$\Pi_1 = \frac{\mathcal{D}}{w^2V^2\rho}$$

Next the procedure is repeated with the second nonrepeating variable, h, so that

$$\Pi_2 = h w^a V^b \rho^c$$

Finally, we can express the results of the dimensional analysis in the form

$$\frac{\mathcal{D}}{w^2V^2\rho} = \tilde{\phi}\left(\frac{h}{w}, \frac{\mu}{wV\rho}\right) \qquad \textbf{(Ans)}$$

It follows that

$$(L)(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

and

$$c = 0 \qquad \text{(for } M\text{)}$$

$$1 + a + b - 3c = 0 \qquad \text{(for } L\text{)}$$

$$b = 0 \qquad \text{(for } T\text{)}$$

so that $a = -1$, $b = 0$, $c = 0$, and therefore

$$\Pi_2 = \frac{h}{w}$$

The remaining nonrepeating variable is μ so that

$$\Pi_3 = \mu w^a V^b \rho^c$$

with

$$(ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

and, therefore,

$$1 + c = 0 \qquad \text{(for } M\text{)}$$

$$-1 + a + b - 3c = 0 \qquad \text{(for } L\text{)}$$

$$-1 - b = 0 \qquad \text{(for } T\text{)}$$

Solving for the exponents, we obtain $a = -1$, $b = -1$, $c = -1$ so that

$$\Pi_3 = \frac{\mu}{wV\rho}$$

Now that we have the three required pi terms we should check to make sure they are dimensionless. To make this check we use F, L, and T, which will also verify the correctness of the original dimensions used for the variables. Thus,

$$\Pi_1 = \frac{\mathfrak{D}}{w^2V^2\rho} \doteq \frac{(F)}{(L)^2(LT^{-1})^2(FL^{-4}T^2)} \doteq F^0L^0T^0$$

$$\Pi_2 = \frac{h}{w} \doteq \frac{(L)}{(L)} \doteq F^0L^0T^0$$

$$\Pi_3 = \frac{\mu}{wV\rho} \doteq \frac{(FL^{-2}T)}{(L)(LT^{-1})(FL^{-4}T^2)} \doteq F^0L^0T^0$$

$$\frac{\mathfrak{D}}{w^2\rho V^2} = \phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right) \qquad \textbf{(Ans)}$$

7.4.1 Selection of Variables

Geometry. The geometric characteristics can usually be described by a series of lengths and angles. In most problems the geometry of the system plays an important role.

Material Properties. Since the response of a system to applied external effects such as forces, pressures, and changes in temperature is dependent on the nature of the materials involved in the system, the material properties that relate the external effects and the responses must be included as variables. For example,

External Effects. This terminology is used to denote any variable that produces, or tends to produce, a change in the system.

In summary, the following points should be considered in the selection of variables:

1. Clearly define the problem. What is the main variable of interest (the dependent variable)?
2. Consider the basic laws that govern the phenomenon. Even a crude theory that describes the essential aspects of the system may be helpful.
3. Start the variable selection process by grouping the variables into three broad classes: geometry, material properties, and external effects.
4. Consider other variables that may not fall into one of the above categories. For example, time will be an important variable if any of the variables are time dependent.
5. Be sure to include all quantities that enter the problem even though some of them may be held constant (e.g., the acceleration of gravity, g). For a dimensional analysis it is the dimensions of the quantities that are important—not specific values!
6. Make sure that all variables are independent. Look for relationships among subsets of the variables.

TABLE 7.1

Some Common Variables and Dimensionless Groups in Fluid Mechanics

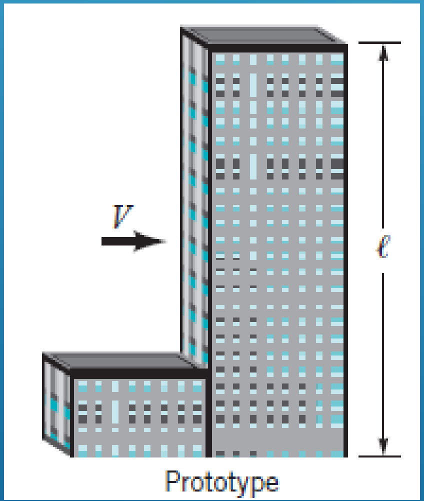
Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

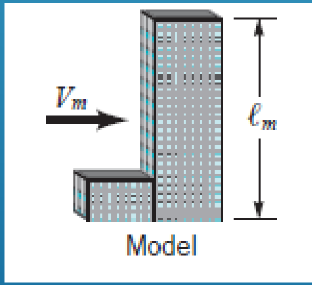
Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number,[a] Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{\omega \ell}{V}$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

7.8 Modeling and Similitude

Models are widely used in fluid mechanics. Major engineering projects involving structures, aircraft, ships, rivers, harbors, dams, air and water pollution, and so on, frequently involve the use of models. Although the term "model" is used in many different contexts, the "engineering model" generally conforms to the following definition. *A* ***model*** *is a representation of a physical system that may be used to predict the behavior of the system in some desired respect.* The physical system for which the predictions are to be made is called the ***prototype.*** Although *mathematical* or *computer* models may also conform to this definition, our interest will be in physical models, that is, models that resemble the prototype but are generally of a different size, may involve different fluids, and often operate under different conditions (pressures, velocities, etc.).

The similarity requirements for a model can be readily obtained with the aid of dimensional analysis.



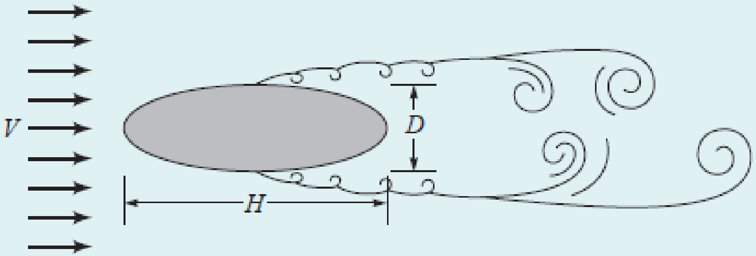


Similarity between a model and a prototype is achieved by equating pi terms.

EXAMPLE 7.5 Prediction of Prototype Performance from Model Data

GIVEN A long structural component of a bridge has an elliptical cross section shown in Fig. E7.5. It is known that when a steady wind blows past this type of bluff body, vortices may develop on the downwind side that are shed in a regular fashion at some definite frequency. Since these vortices can create harmful periodic forces acting on the structure, it is important to determine the shedding frequency. For the specific structure of interest, $D = 0.1$ m, $H = 0.3$ m, and a representative wind velocity is 50 km/hr. Standard air can be assumed. The shedding frequency is to be determined through the use of a small-scale model that is to be tested in a water tunnel. For the model $D_m = 20$ mm and the water temperature is 20 °C.

FIND Determine the model dimension, H_m, and the velocity at which the test should be performed. If the shedding frequency for the model is found to be 49.9 Hz, what is the corresponding frequency for the prototype?



■ FIGURE E7.5

Solution

We expect the shedding frequency, ω, to depend on the lengths D and H, the approach velocity, V, and the fluid density, ρ, and viscosity, μ. Thus,

$$\omega = f(D, H, V, \rho, \mu)$$

where

$$\omega \doteq T^{-1}$$
$$D \doteq L$$
$$H \doteq L$$
$$V \doteq LT^{-1}$$
$$\rho \doteq ML^{-3}$$
$$\mu \doteq ML^{-1}T^{-1}$$

Since there are six variables and three reference dimensions (MLT), three pi terms are required. Application of the pi theorem yields

$$\frac{\omega D}{V} = \phi\left(\frac{D}{H}, \frac{\rho V D}{\mu}\right)$$

We recognize the pi term on the left as the Strouhal number, and the dimensional analysis indicates that the Strouhal number is a function of the geometric parameter, D/H, and the Reynolds number. Thus, to maintain similarity between model and prototype

$$\frac{D_m}{H_m} = \frac{D}{H}$$

and

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho V D}{\mu}$$

From the first similarity requirement

$$H_m = \frac{D_m}{D} H$$

$$= \frac{(20 \times 10^{-3}\ \text{m})}{(0.1\ \text{m})}(0.3\ \text{m})$$

$$H_m = 60 \times 10^{-3}\ \text{m} = 60\ \text{mm} \qquad \textbf{(Ans)}$$

The second similarity requirement indicates that the Reynolds number must be the same for model and prototype so that the model velocity must satisfy the condition

$$V_m = \frac{\mu_m}{\mu}\frac{\rho}{\rho_m}\frac{D}{D_m} V \qquad \textbf{(1)}$$

For air at standard conditions, $\mu = 1.79 \times 10^{-5}\ \text{kg/m} \cdot \text{s}$, $\rho = 1.23\ \text{kg/m}^3$, and for water at 20 °C, $\mu = 1.00 \times 10^{-3}\ \text{kg/m} \cdot \text{s}$, $\rho = 998\ \text{kg/m}^3$. The fluid velocity for the prototype is

$$V = \frac{(50 \times 10^3\ \text{m/hr})}{(3600\ \text{s/hr})} = 13.9\ \text{m/s}$$

The required velocity can now be calculated from Eq. 1 as

$$V_m = \frac{[1.00 \times 10^{-3}\ \mathrm{kg/(m \cdot s)}]}{[1.79 \times 10^{-5}\ \mathrm{kg/(m \cdot s)}]} \frac{(1.23\ \mathrm{kg/m^3})}{(998\ \mathrm{kg/m^3})} \times \frac{(0.1\ \mathrm{m})}{(20 \times 10^{-3}\ \mathrm{m})}(13.9\ \mathrm{m/s})$$

$$V_m = 4.79\ \mathrm{m/s}$$ **(Ans)**

This is a reasonable velocity that could be readily achieved in a water tunnel.

With the two similarity requirements satisfied, it follows that the Strouhal numbers for prototype and model will be the same so that

$$\frac{\omega D}{V} = \frac{\omega_m D_m}{V_m}$$

and the predicted prototype vortex shedding frequency is

$$\omega = \frac{V}{V_m}\frac{D_m}{D}\omega_m = \frac{(13.9\ \mathrm{m/s})}{(4.79\ \mathrm{m/s})}\frac{(20 \times 10^{-3}\ \mathrm{m})}{(0.1\ \mathrm{m})}(49.9\ \mathrm{Hz})$$

$$\omega = 29.0\ \mathrm{Hz}$$ **(Ans)**

7.8.2 Model Scales

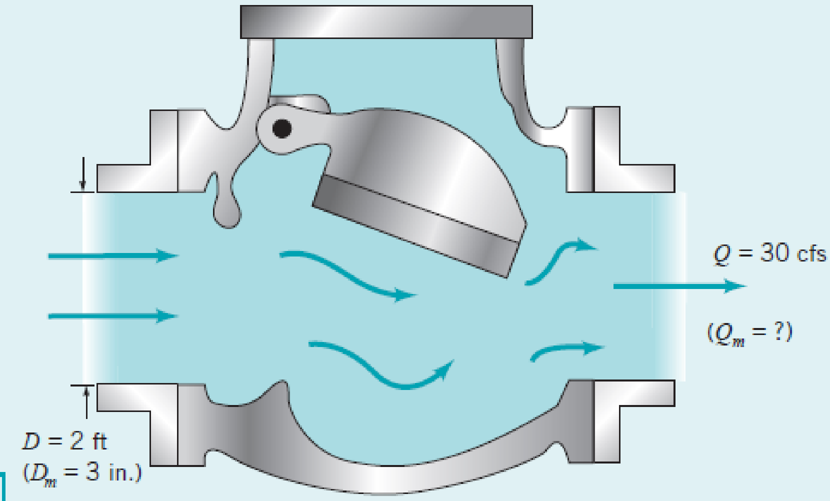
It is clear from the preceding section that the ratio of like quantities for the model and prototype naturally arises from the similarity requirements. For example, if in a given problem there are two length variables ℓ_1 and ℓ_2, the resulting similarity requirement based on a pi term obtained from these two variables is

$$\frac{\ell_1}{\ell_2} = \frac{\ell_{1m}}{\ell_{2m}}$$

Example 7.6 Reynolds Number Similarity

GIVEN Model tests are to be performed to study the flow through a large check valve having a 2-ft-diameter inlet and carrying water at a flowrate of 30 cfs as shown in Fig. E7.6*a*. The working fluid in the model is water at the same temperature as that in the prototype. Complete geometric similarity exists between model and prototype, and the model inlet diameter is 3 in.

FIND Determine the required flowrate in the model.



■ FIGURE E7.6*a*

Solution

To ensure dynamic similarity, the model tests should be run so that

$$\mathrm{Re}_m = \mathrm{Re}$$

or

$$\frac{V_m D_m}{\nu_m} = \frac{VD}{\nu}$$

where V and D correspond to the inlet velocity and diameter, respectively. Since the same fluid is to be used in model and prototype, $\nu = \nu_m$, and therefore

$$\frac{V_m}{V} = \frac{D}{D_m}$$

The discharge, Q, is equal to VA, where A is the inlet area, so

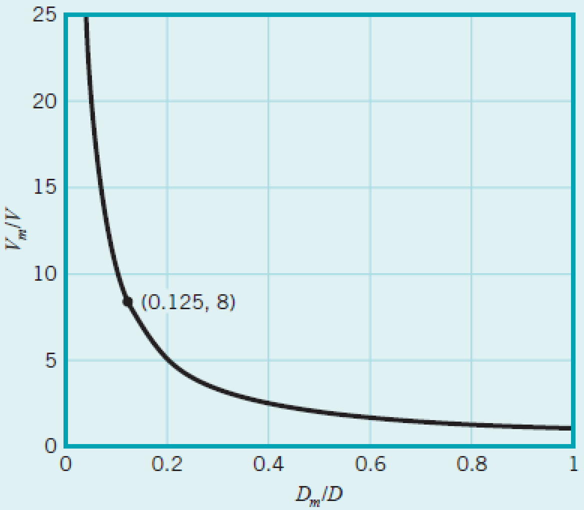
$$\frac{Q_m}{Q} = \frac{V_m A_m}{VA} = \left(\frac{D}{D_m}\right)\frac{[(\pi/4)D_m^2]}{[(\pi/4)D^2]}$$

$$= \frac{D_m}{D}$$

and for the data given

$$Q_m = \frac{(3/12 \text{ ft})}{(2 \text{ ft})}(30 \text{ ft}^3/\text{s})$$

$$Q_m = 3.75 \text{ cfs}$$ **(Ans)**



■ FIGURE E7.6*b*

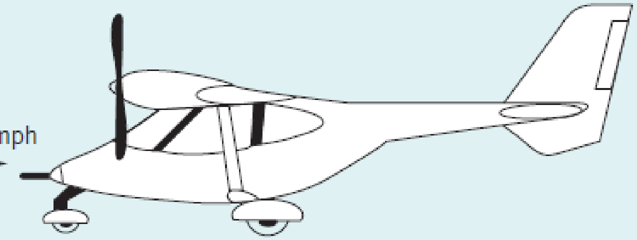
EXAMPLE 7.7 Model Design Conditions and Predicted Prototype Performance

GIVEN The drag on the airplane shown in Fig. E7.7 cruising at 240 mph in standard air is to be determined from tests on a 1:10 scale model placed in a pressurized wind tunnel. To minimize compressibility effects, the air speed in the wind tunnel is also to be 240 mph.

FIND Determine

(a) the required air pressure in the tunnel (assuming the same air temperature for model and prototype) and

(b) the drag on the prototype corresponding to a measured force of 1 lb on the model.



■ FIGURE E7.7

SOLUTION

(a) From Eq. 7.19 it follows that drag can be predicted from a geometrically similar model if the Reynolds numbers in model and prototype are the same. Thus,

$$\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu}$$

For this example, $V_m = V$ and $\ell_m/\ell = \frac{1}{10}$ so that

$$\frac{\rho_m}{\rho} = \frac{\mu_m}{\mu}\frac{V}{V_m}\frac{\ell}{\ell_m}$$

$$= \frac{\mu_m}{\mu}(1)(10)$$

and therefore

$$\frac{\rho_m}{\rho} = 10\frac{\mu_m}{\mu}$$

This result shows that the same fluid with $\rho_m = \rho$ and $\mu_m = \mu$ cannot be used if Reynolds number similarity is to be maintained. One possibility is to pressurize the wind tunnel to increase the density of the air. We assume that an increase in pressure does not significantly change the viscosity so that the required increase in density is given by the relationship

$$\frac{\rho_m}{\rho} = 10$$

For an ideal gas, $p = \rho RT$ so that

$$\frac{p_m}{p} = \frac{\rho_m}{\rho}$$

for constant temperature ($T = T_m$). Therefore, the wind tunnel would need to be pressurized so that

$$\frac{p_m}{p} = 10$$

Since the prototype operates at standard atmospheric pressure, the required pressure in the wind tunnel is 10 atmospheres or

$$\begin{aligned} p_m &= 10\,(14.7\text{ psia}) \\ &= 147\text{ psia} \end{aligned}$$

(Ans)

(b) The drag could be obtained from Eq. 7.19 so that

$$\frac{\mathcal{D}}{\frac{1}{2}\rho V^2 \ell^2} = \frac{\mathcal{D}_m}{\frac{1}{2}\rho_m V_m^2 \ell_m^2}$$

or

$$\begin{aligned} \mathcal{D} &= \frac{\rho}{\rho_m}\left(\frac{V}{V_m}\right)^2\left(\frac{\ell}{\ell_m}\right)^2 \mathcal{D}_m \\ &= \left(\frac{1}{10}\right)(1)^2(10)^2\mathcal{D}_m \\ &= 10\mathcal{D}_m \end{aligned}$$

Thus, for a drag of 1 lb on the model the corresponding drag on the prototype is

$$\mathcal{D} = 10\text{ lb}$$

(Ans)