Chapter 4: Nominal and Effective Interest Rate

Nominal interest rate, **r**, is an interest rate that does not include any consideration of compounding. By definition,

r = interest rate per period x number of periods.

e.g. r = 1% per month, or 3% per quarter (three months in a quarter), or 6% per six-month, or 12% per year, or 24% per two-year.

Effective interest rate is the actual rate that applies for a stated period of time. The compounding during the time period of the corresponding nominal rate is accounted for by the effective interest rate.

An effective rate has the compounding frequency attached to the nominal rate, e.g. 12% per year, compounded monthly. This indicates that the nominal rate is 12% per year and effective rate is the rate that would result by accounting for the compounding of interest every month. If on the other hand the interest rate was quoted as 12% per year compounded quarterly, the nominal rate would still be 12% per year with the effective rate determined by considering compounding every quarter, i.e. every 3-month.

The general formula to determine effective "i" is given by,

$$\textbf{Effective i/period} = (1 + r/m)^m - 1$$

where r is the nominal rate/period, and "m" is the number of compounding periods per period. If we wish to find the effective interest rate/year, then "**r**" **has** to be the nominal rate per year and "m" will be the number of compounding periods/year.

For example, we wish to determine (a) effective i/year, and (b) i/quarter, for the interest rate of 12% per year compounded quarterly.

(a) Since we wish to find i/year, then we have to use r/year (=12%) and number of compounding periods in one year for "m" (= 12/3 = 4).

Then, $i = (1 + 0.12/4)^4 - 1 = 0.1255$ (12.55% per year)

(b) For i/quarter, we need to use nominal rate per quarter, i.e. 0.12/4 = 0.03, and m = 1 (only one compounding in one quarter since compounding period is quarter).

Then, $i = (1 + 0.03/1)^1 - 1 = 0.03$

Note that the effective rate/compounding period = nominal rate/compounding period.

The following tables give some effective interest rates for given nominal rates and compounding periods.

Interest rate = 18% per year **compounded monthly**

Interest period	**Nominal, r%**	**Effective, i%**	**m**
per month	**1.5**	**1.5**	**1**
per quarter	4.5	4.57	3
per year	18	19.56	12
per two-year	36	42.95	24

Interest rate = 8% per year compounded **semiannually**

per month	0.667	-	<1
per quarter	2	-	<1
per six-month	**4**	**4**	**1**
per year	8	8.16	2
per two-year	16	16.99	4

Note: effective rate calculations for cases where m<1, i.e. compounding period is longer than the period considered, are not carried out as these cases will not be studied within this course.

Also note that when interest rate is quoted as, for example, 5% per year with no mention of compounding, then 5% is the effective rate per year. Similarly, 3% per quarter would indicate effective rate per quarter.

To determine the effective rate per year in the case of 3% per quarter, we have to rely on the above formula:

Nominal rate/year = (0.03).(4) = 0.12 or 12%; m = 4 compounding (4 quarters in a year)

$$i/\text{year} = (1 + 0.12/4)^4 - 1 = 0.1255 \quad (12.55\% \text{ per year})$$

Additional examples:

1. Express the interest rate of 6.5% per semiannual period compounded weekly in the following forms.

(a) effective rate per week
(b) effective yearly rate
(c) effective rate per semiannual period
(d) effective rate per quarter

(a) since compounding is weekly, effective rate is the same as nominal rate,

weekly i = 0.065/26 = 0.0025 or 0.25% per week (26 weeks in half year)

(b) yearly $i = [1 + (0.065\text{x}2)/52]^{52} - 1 = 0.1386$ or 13.86% per year

(c) semiannual $i = (1 + 0.065/26)^{26} - 1 = 0.067$ or 6.7% per semiannual

(d) quarterly $i = [1 + (0.065/2)/13]^{13} - 1 = 0.03299$ or 3.299% per quarter

2. (a) An effective yearly rate is quoted at 12% per year where the compounding period is monthly. What is the effective monthly rate?
(b) Find the nominal interest rate per year and effective interest rate per compounding period for an effective annual interest rate of 12.36% where compounding is semiannual.

(a) monthly i = monthly nominal rate since compounding is monthly.

Then, $0.12 = (1 + r/12)^{12} – 1$ where r is the nominal rate/year
or, monthly $i = r/12 = (1.12)^{1/12} – 1 = 0.00949$ or 0.949% per month.

(b) Compounding period is semiannual. Then, semiannual i = semiannual nominal rate.
Let r = nominal rate/year.
Then, $0.1236 = (1 + r/2)^2 - 1$
or, semiannual $i = r/2 = (1.1236)^{1/2} – 1 = 0.06$ or 6% semiannually
Nominal rate/year = (0.06).2 = 0.12 or 12% per year.

Continuous Compounding

When compounding is continuous, the relationship between the effective i and nominal rate is given by,

$$i/period = e^r -1$$

where r is the nominal rate per period.

Example:

3. For an interest rate of 18% per year, compounded continuously, calculate the effective monthly, quarterly and annual interest rates.

For monthly rate calculation we need to use the monthly nominal rate that is,

$$= 0.18/12 = 0.015$$

Then, $i/month = e^{0.015} – 1 = 0,01511$ or 1.511% per month.

For quarterly rate, r/quarter = 0.18/4 = 0.045
Then,
$i/quarter = e^{0.045} –1 = 0.046$ or 4.6% per quarter.

For annual rate, r/year = 0.18
Then,
$i/year = e^{0.18} – 1 = 0.1972$ or 19.72% per year.

It is to be reminded again that in calculations involving economic values, i.e. where we determine present or future values, etc., we always use the effective interest rates. The effective rate that we would find most convenient to use will depend on the type and frequency of payments involved. The following provide some examples:

Equivalence Relations: Single Amounts with Payment Period, PP ≥ CP, Compounding Period

When only single-amount cash flows are involved, we could use either of the following two methods to determine i and n for P/F and F/P.

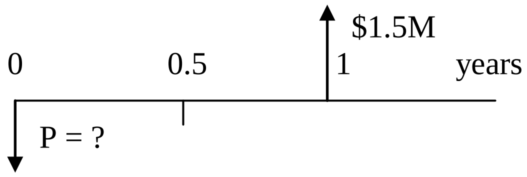
Method 1: Determine the effective interest rate over the compounding period CP, and set n equal to the number of compounding periods between P and F.

Method 2: Determine the effective interest rate for the time period t of the nominal rate and set n equal to the number of periods using this same time period. (or, alternatively, determine the effective i for the payment period and use this period to determine n).

Examples:

4. A new mobile phone company estimates that by advertising its new model, it will increase its sales by \$1.5 million one year from now when the phones go on sale. At an interest rate of 20% per year, compounded semi-annually, what is the maximum amount the company can afford to spend now on advertising in order to break even?

i = 20% per year compounded semiannually



If we are to use Method 1, we first determine the effective rate per compounding period. In this case, i = 10% per six-months (CP = 6 months). Then, n = 2, i.e. 2 six-month periods between P and F.

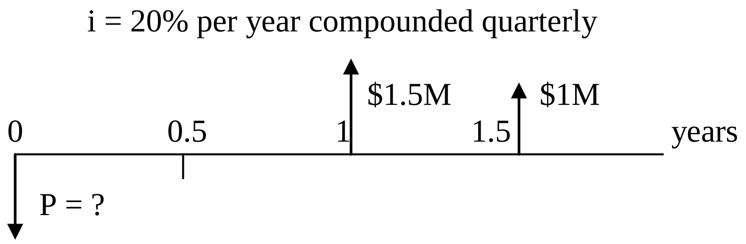
Then, $P = 1.5(P/F,10\%,2)$
$= 1.5(0.8264)$
$= \$1.2396$ million

If we are to use Method 2, we first determine the effective i per year (nominal rate period). Then, n = 1, i.e. one year between P and F.

$$\text{effective i/year} = (1 + 0.2/2)^2 - 1 = 0.21 \qquad \text{or } 21\%$$

and, $P = 1.5(P/F,21\%,1)$
$= 1.5(0.8264)$
$= \$1.2396$ million.

5. A new company estimates that by investing in a new process, it will increase its sales by \$1.5 million one year from now and a further \$1 million one and half year from now. At an interest rate of 20% per year, compounded quarterly, what is the maximum amount the company can afford to spend now on the new process in order to break even?



If we are to use Method 1, we first determine the effective rate per compounding period. In this case, i = 5% per quarter (CP = 3 months). Then, n is measured in terms of the compounding period i.e. quarter or three-month periods.

Then,
$$P = 1.5(P/F,5\%,4) + 1.(P/F,5\%,6)$$
$$= 1.5(0.8227) + 1.(0.7462)$$
$$= \$1.98025 \text{ million}$$

If we are to use Method 2, and we decide to use the nominal rate period, we must first determine the effective i per year. It is now to be noted that the value of n will be 1.5 for the sales increase a year and half from now. The n=1.5 can still be used in the formulas to determine P. Then,

$$i/\text{year} = (1 + 0.2/4)^4 - 1 = 0.2155 \qquad \text{or } 21.55\% \text{ per year.}$$

and,
$$P = 1.5(P/F,21.55\%,1) + 1.(P/F,21.55\%,1.5)$$
$$= 1.5(0.8227) + 1.(0.7462)$$
$$= \$1.98025 \text{ million.}$$

Alternatively, we may use the effective interest rate per payment period that is six months. the value of n will then be measured in six-month periods.

Then,
$$i/\text{six-month} = (1 + 0.1/2)^2 - 1 = 0.1025 \qquad \text{or } 10.25\% \text{ per six-month}$$

and,
$$P = 1.5.(P/F,10.25\%,2) + 1.(P/F,10.25\%,3)$$
$$= 1.5(0.8227) + 1.(0.7462)$$
$$= \$1.98025$$

5. Ali makes deposits of \$10000 at the end of year 2, \$25000 at the end of year 3, and \$30000 at the end of year 5. Determine the future worth (in year 5) of the deposits at an interest rate of 16% per year, compounded semiannually.

Method 1: Using the effective i per CP (= 6 months in this case). n will then be measured in terms of CP.

i/six-months = 16/2 = 8%

Then,

F = 10,000(F/P,8%,6) + 25,000(F/P,8%,4) + 30,000
= 10,000(1.5869) + 25,000(1.3605) + 30,000
= \$79,881

or

Method 2: Using the effective i per nominal rate period, i.e. i/year,

effective i/year = $(1 + 0.16/2)^2 - 1$ = 16.64%
F = 10,000(F/P,16.64%,3) + 25,000(F/P,16.64%,2) + 30,000
= \$79,881

Equivalence Relations: Series with PP ≥ CP

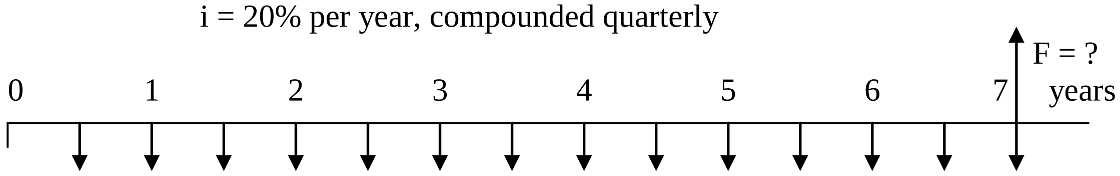
When cash flows involve a series (i.e., A, G, g) and PP ≥ CP,

- Find the effective interest rate **i** per payment period.
- Determine **n** as the total number of payment periods.

Examples:

6. Mehmet deposited \$1000 every six months into his account for the past 7 years. If the interest rate was 20% per year, compounded quarterly, how much does he have in his account today, having just made his last deposit?

i = 20% per year, compounded quarterly



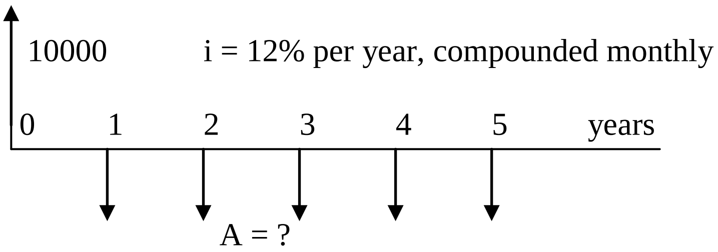
Payment period = six months. We, therefore, have to use the effective i/six-months and n will have to be the total number of payment periods, i.e. number of six-month periods in 7 years (=14).

The nominal rate/six-months = 10%
Number of compoundings in six-months = 2 (every quarter)

Therefore, effective i = (1 + 0.1/2)2 – 1 = 0.1025 or 10.25%

Then, F = A(A/F,10.25%,14) = 1000(28.4891)
= \$28489.1

7. I borrowed \$10000 from the bank that charges 12% per year, compounded monthly interest. I wish to repay the loan by making 5 annual payments with the first payment one year from now. What will be my annual payments?

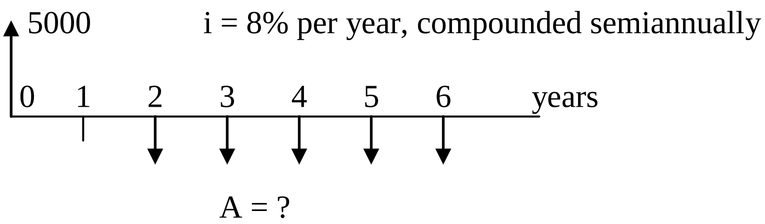


Since payments will be annual, we have to use the effective rate per year and n will be the number of payment periods, i.e. 5 (years).

$$i/\text{year} = (1 + 0.12/12)^{12} - 1 = 0.1268 \quad \text{or } 12.68\%$$

Then, $A = 10000(A/P, 12.68\%, 5) = 10000(0.282)$
$= \$2820$ per year.

8. An individual borrows \$5000 at an interest rate of 8% per year compounded semiannually and desires to repay the money with five equal end-of-year payments, with the first payment made two years after receiving the \$5000. (a) What should be the size of the annual payment? (b) At the time of the fourth payment, suppose he decides to pay off the loan with one lump sum payment. How much should be paid?



Payments are annual; we, therefore, have to use effective interest rate per year in A/P or P/A factors. Then,

$$i/\text{year} = (1 + 0.08/2)^2 - 1 = 0.0816 \quad \text{or } 8.16\%$$

a) $5000 = A(P/A,8.16\%,5)(P/F,8.16\%,1)$
$5000 = Ax3.9762x0.9245$
$A = 1360.09$

b) At fourth payment, i.e. fifth year, he has to pay the amount A and the value of the sixth payment at the fifth year:

$$= A + A(P/F,8.16\%,1) = 1360.09 + 1360.09(0.9245)$$
$$= 2617.57$$

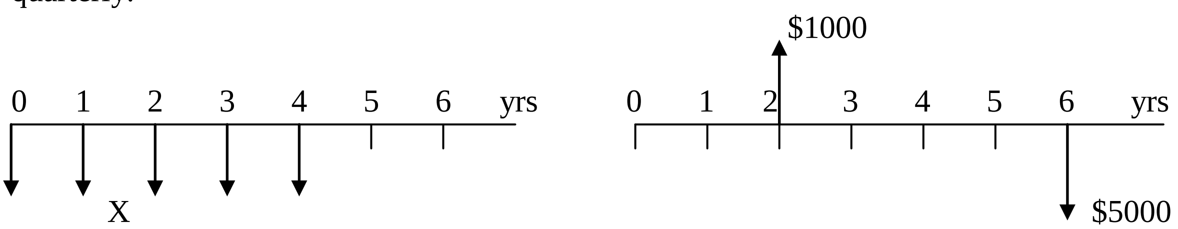
Alternatively, he has to pay the difference between the future value of 5000 at the fifth year and the future value of the payments he has made up to the fifth year. Then, he has to pay:

(i) 5000(F/P,8.16%,5) – A(F/A,8.16%,3)(F/P,8.16%,1)
= 5000(1.4802) –1360.09(3.251)(1.0816) = 2618 or, formulating differently,

(ii) 5000(F/P,8.16%,5) – [A(F/A,8.16%,4) - A] = 7401.22 – 1360.09(4.5168) + 1360.09
= 2618

Note: In (ii) above, A is subtracted since A(F/A,8.16%,4) value assumes that fifth payment has already been made.

9. Determine the value of X on the left-hand cash flow diagram that establishes equivalence with the right-hand cash flow diagram. The interest rate is 12% per year compounded quarterly.



The series of payment are annual; interest rate to be used must be annual. Then,

$$i/year = (1 + 0.12/4)^4 – 1 = 0.1255 \quad (12.55\%)$$

The cash flow amounts on the right are single amounts. For these, we can use the interest rate/CP or interest rate/nominal rate period, i.e. annual.

Then, using annual rates for both cash flows and since they are equivalent,

-X – X.(P/A,12.55%,4) = 1000.(P/F,12.55%,2) – 5000.(P/F,12.55%,6)
-X – 3.002X = 1000.(0.78942) – 5000.(0.49195) = 789.42 – 2459.78 = - 1670.36
X = $417.38

Similarly, we could have written:

– X.(P/A,12.55%,5) = 1000.(P/F,12.55%,3) – 5000.(P/F,12.55%,7)
X = $417.38

We can also use interest rate/CP, i.e. per quarter, for the single amounts. Then,
i/quarter = 3%,

-X – X.(P/A,<u>12.55%</u>,4) = 1000.(P/F,<u>3%</u>,8) – 5000.(P/F,<u>3%</u>,24) or

– X.(P/A,12.55%,5) = 1000.(P/F,3%,12) – 5000.(P/F,3%,28)

X = $417.38

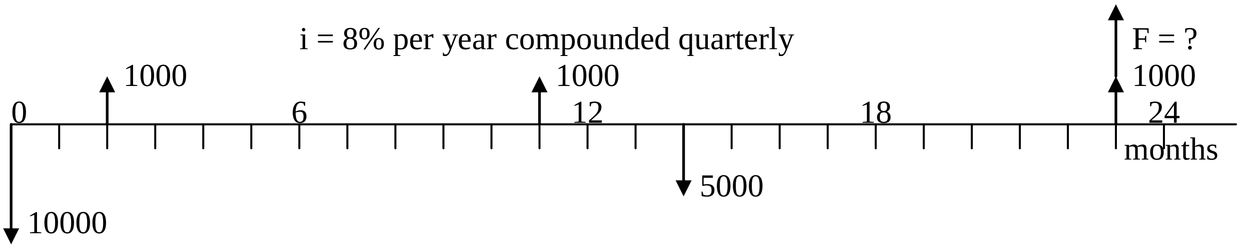
Equivalence Relations: Single amounts and Series with PP < CP

In this section, only the cases where there is no inter-period interest payment or inter-period compounding will be presented. Then,

- Deposits (negative cash flows) are all regarded as deposited at the end of the compounding period, and
- Withdrawals (positive cash flows) are all regarded as withdrawn at the beginning of the compounding period.

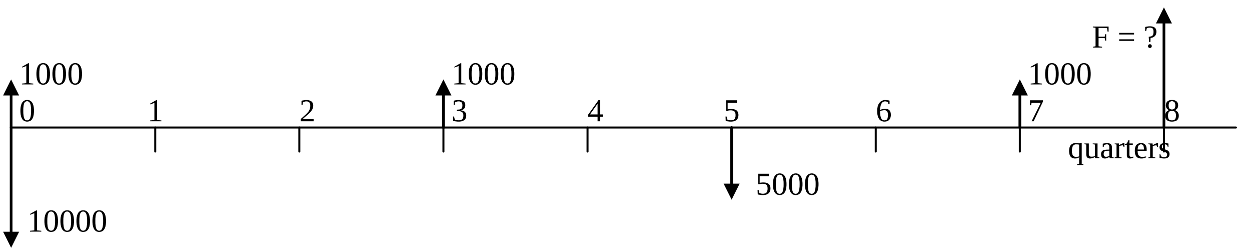
Examples:

10. I deposit $10000 into my account which pays interest at 8% per year, compounded quarterly. If I withdraw $1000 in months 2, 11, and 23 from now, and make an additional deposit of $5000 in month 14 from now, how much would I have in my account at the end of two years? Assume no inter-period compounding.



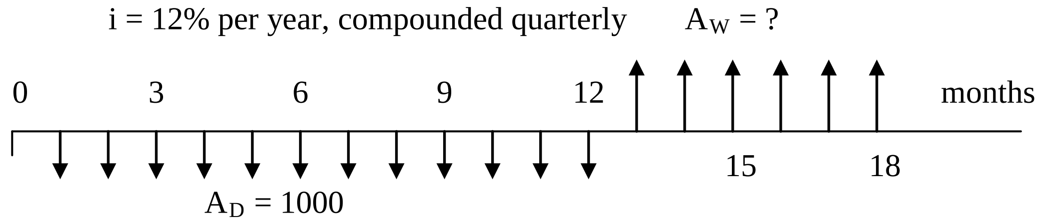
Since no inter-period compounding, cash flow modifies to the following with the interest rate at 2% per quarter.

Move $1000 withdrawal in month 2 to time 0. Move $1,000 withdrawal in month 11 to month 9 (quarter 3). Move $1000 withdrawal in month 23 to month 21 (quarter 7). Move $5000 deposit in month 14 to month 15 (quarter 5).

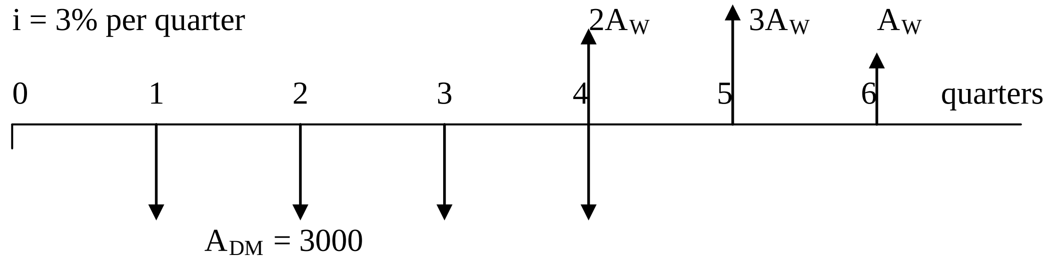


$$
\begin{aligned}
F &= (10{,}000 - 1000)(F/P,2\%,8) - 1000(F/P,2\%,5) - 1000(F/P,2\%,1) + 5000(F/P,2\%,3) \\
&= 9000(1.1717) - 1000(1.1041) - 1000(1.0200) + 5000(1.0612) \\
&= \$13727.2
\end{aligned}
$$

11. Mehmet has been saving money by making deposits of \$1000 per month for the past year. He has now decided to withdraw equal amounts from his savings every month, starting one month from now. If he wishes to withdraw all his money in six months and the interest rate is 12% per year compounded quarterly, what will be the size of his withdrawals? Assume that there is no inter-period compounding.



Since there is no inter-period interest payment, cash flow modifies to:



Considering the values at quarter 4:

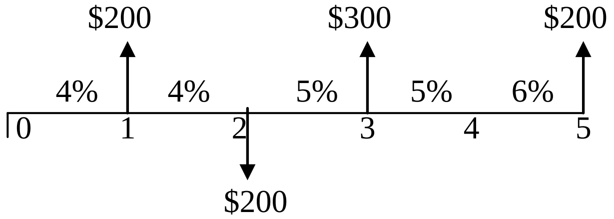
$$A_{DM}(F/A,3\%,4) = 2A_W + 3A_W(P/F,3\%,1) + A_W(P/F,3\%,2)$$
$$3000(4.1836) = 2A_W + 3A_W(0.9709) + A_W(0.9426)$$
$$12550.8 = 5.8553A_W$$
$$A_W = 2143.49$$

Interest Rates that Vary Over Time

The applicable interest rate is used for each period when calculating present, annual, or future values.

Examples:

12. Consider the cash flow given below with the appropriate interest rates indicated. Determine its (a) present worth, and (b) future worth.



(a) PW = 200(P/F,4%,1) – 200(P/F,4%,2) + 300(P/F,5%,1)(P/F,4%,2)

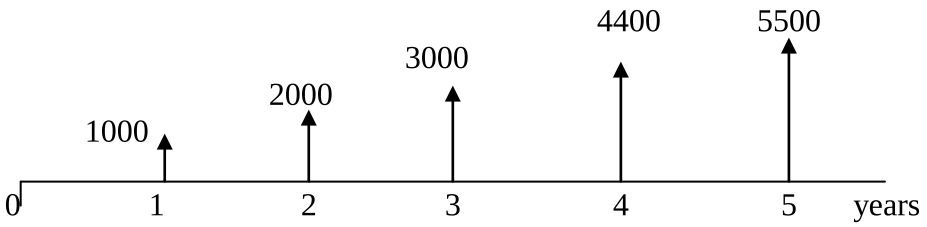
+ 200(P/F,6%,1)(P/F,5%,2)(P/F,4%,2)
= $429.78

(b) FW = 200 + 300(F/P,5%,1)(F/P,6%,1) – 200(F/P,5%,2)(F/P,6%,1)
+ 200(F/P,4%,1)(F/P,5%,2)(F/P,6%,1)
= $543.25

or

FW = PW(F/P,4%,2)(F/P,5%,2)(F/P,6%,1)
= $543.25

13. Consider the cash flow diagram shown below. Determine its (a) present worth and (b) its equivalent uniform end-of-year series.

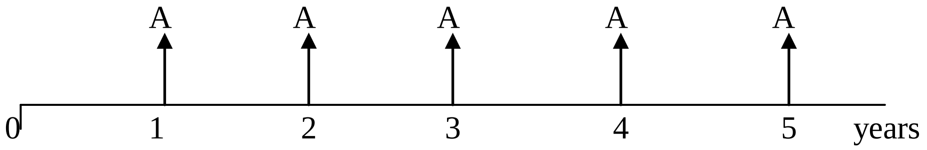


Interest rate for years 1 and 2 ------ %0.5 per month
Interest rate for year 3 ----------- % 6.5 per year
Interest rate for years 4 and 5 ----- % 3 per semiannual period

P = 100(P/F,%0.5,12) + 2000(P/F,%0.5,24) + 3000(P/F,%6.5,1)(P/F,%0.5,24)
+ 4400(P/F,%3,2)(P/F,%6.5,1)(P/F,%0.5,24) + 5500(P/F,%3,4)(P/F,%6.5,1)(P/F,%0.5,24)

P = 1000(0.9419) + 2000(0.8872) + 3000(0.8330) + 4400(0.78522) + 5500(0.7401)
= 12740.94

(b) The equivalent uniform end-of-year series that is to be determined is as follows:



Then,

P = A(P/F,%0.5,12) + A(P/F,%0.5,24) + A(P/F,%6.5,1)(P/F,%0.5,24)
+ A(P/F,%3,2)(P/F,%6.5,1)(P/F,%0.5,24) + A(P/F,%3,4)(P/F,%6.5,1)(P/F,%0.5,24)

12740.94 = A(0.9419+0.8872+0.83304+0.78522+0.7401)

4.1875A=12740.94

A=3042.67 $