Chapter 2: How Time and Interest Affect Money

The value of money is a function of time and interest rate. The general relationship that can be applied to determine the time value of money at various times is in the form:

X = Y.(X/Y, i , n)

Where, X is the value to be determined, Y is the known or given value, and (X/Y, i , n) is a function of (i,n), often referred to as (X/Y) factor.

In line with the above, if we wish to determine the future value (F) of certain present amount of money, i.e. present value or present worth = P, we determine F from,

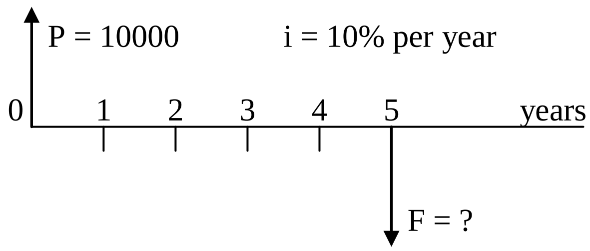
F = P.(F/P, i , n)

The function (F/P, i , n), also referred to as F/P factor, is given by $(1 + i)^n$.

Similar relationships can be written for the relationships between P, F, and A where A represents uniform series of end-of-period cash flow. The relevant formulas for the factors are given in the Engineering Economy textbooks along with the tabulated values for various interest rates (i) and interest periods (n). It is to be noted that the formulation related to A is relevant to uniform series of cash flow starting at the end of the first interest period, i.e. starting at year one when interest rate is per year basis.

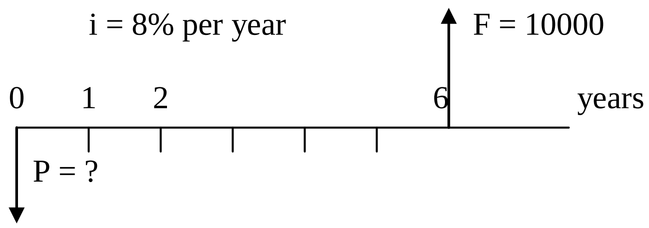
Examples:

1. I borrow $10000 at 10% per year interest. I plan to repay the loan after 5 years. How much would I pay then? (i.e. what is the future value of $10000 at year 5 given i and n?)



F = P. (F/P, i , n) = 10000.(F/P, 10%, 5) = 10000.(1.6105)
= $16105

2. I plan to have $10000 in my bank account in 6 years time. If the interest rate is 8% per year, how much do I have to deposit now? (i.e. what is the present value of the $10000 given i and n?)



$P = F.(P/F, i, n) = 10000.(P/F, 8\%, 6) = 10000.(0.6302)$
$= \$6302$

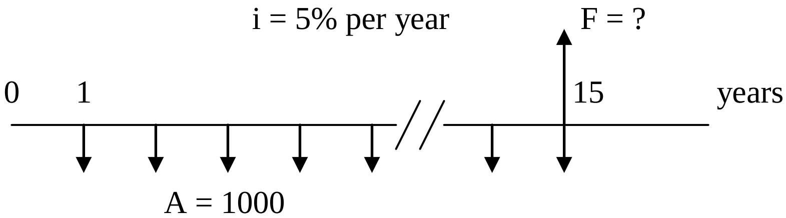
3. I plan to make investments of $600 now, $300 in two years time, and $400 in five years time. If the ROR (rate of return) is expected to be 5% per year, what will be the worth of my investments in ten years time?

$F = 600.(F/P,5\%,10) + 300.(F/P,5\%,8) + 400.(F/P,5\%,5)$
$= 600.(1.6289) + 300.(1.4775) + 400.(1.2763)$
$= 1931.11$

4. A company expects to spend $6000 two years from now, $9000 three years from now, and $5000 six years from now. What is the present worth of the planned expenditures at an interest rate of 10% per year? (Alternative wording: How much money has to be put aside now by the company in order to make these expenditures).

$P = 6000.(P/F,10\%,2) + 9000.(P/F,10\%,3) + 5000.(P/F,10\%,6)$
$= \$14543$

5. I plan to deposit $1000 per year into my account that pays 5% per year interest. The first payment will be made one year from today. How much money will accumulate in my account immediately after the 15th deposit? (i.e. What is the future equivalent or future value of these deposits given i and n?)



$F = 1000.(F/A,5\%,15) = 1000.(21.5786)$
$= \$21578.60$

6. Ali wishes to save $100000 for his retirement by making equal annual deposits into his account that pays 7% per year interest. He is now 20 years old and wishes to retire at 65. How much should he deposit every year? (i.e. What is the annual equivalent or annual worth of the future amount $100000 given i and n?)

$A = 100000.(A/F,7\%,45) = 100000.(0.0035)$
$= \$350$ per year.

7. Mehmet purchased a car for $8000 for which he took a loan from the bank that charges 10% per year interest. If Mehmet wishes to pay back the loan with four equal annual payments, starting one year from now, what will be the size of his payments? (i.e. What is the annual equivalent or annual worth of the present amount $8000 given i and n?)

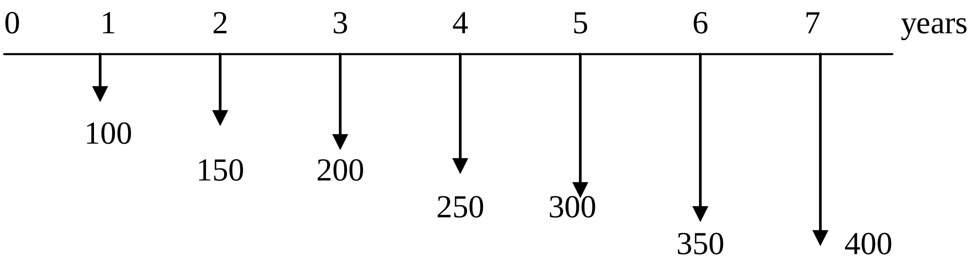
$A = 8000.(A/P,10\%,4) = 8000.(0.3155)$
$= \$2524$ per year.

8. If Company X invests into the upgrade of the production machines, it can improve cash flow by \$20000 at the end of each year for five years. If interest rate is 15% per year, how much can Company X afford to invest into the upgrade of the machines? (i.e. What is the present equivalent or present value of these cash flow improvements given i and n?)

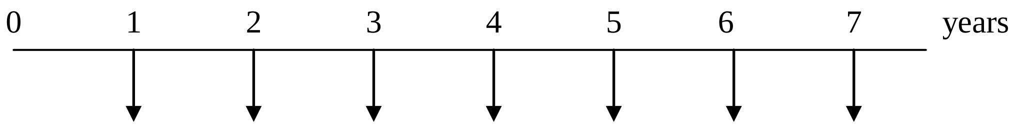
$P = 20000.(P/A,15\%,5) = 20000.(3.3522)$
$= \$67044$

Arithmetic Gradient Cash Flow

An arithmetic gradient is a cash flow series that either increases or decreases by a constant amount. An example is given below.

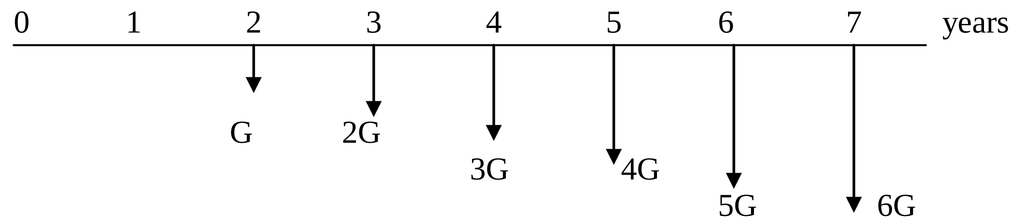


The above cash flow can be represented by,



Base amount, $A_A = 100$

+



Gradient Amount, G = 50

The general equations for calculating total present worth P_T of conventional arithmetic gradients are:

$P_T = P_A + P_G$ for increasing gradients,
$P_T = P_A - P_G$ for decreasing gradients,

where P_A is the present worth of the base amount, and P_G is the present worth of the gradient amount.

Similarly, the equivalent total annual series A_T or total annual worth are given by,

$A_T = A_A + A_G$ for increasing gradients,
$A_T = A_A - A_G$ for decreasing gradients,

where A_A is the annual base amount and A_G is the equivalent annual amount of the gradient series.
(Note: Similar relationships can also be written for the total future values)

In line with the above general formulation,

$P_A = A_A.(P/A,i,n)$ and $P_G = G.(P/G,i,n)$

A_A = as given and $A_G = G.(A/G,i,n)$

Note: The above is for conventional cash flows where, as in example above, uniform series starts at the end of the first interest period, i.e. year one in this example, and gradient starts at the second interest period, i.e. year two in this example.

Examples:

9. Consider the cash flow above, at the beginning of this section. Determine (a) the present worth, and (b) equivalent annual series for an interest rate of 8% per year.

$A_A = 100$ $G = 50$

(a) $P_A = 100.(P/A,8\%,7) = 100.(5.2064) = 520.64$
$P_G = 50.(P/G,8\%,7) = 50.(14.0242) = 701.21$
$P_T = P_A + P_G = 520.64 + 701.21$
$= 1221.85$

(b) $A_G = 50.(A/G,8\%,7) = 50.(2.6937) = 134.685$
$A_T = AA + AG = 100 + 134.685$
$= 234.685$

Note: If the present value is known, as determined in (a), this value can also be used to determine A_T.

$A_T = P_T.(A/P,8\%,7) = 1221.85(0.19207)$
$= 234.68$
Also note that, the future value can also be determined from the value of P_T or A_T.

10. The owner of a new company expects to spend \$1 million the first year with amounts decreasing by \$100000 each year. Income is expected to be \$4 million the first year, increasing by \$500000 each year. Determine the present worth of the new company's net cash flow over a 5-year period at an interest rate of 15% per year.

Here we have decreasing gradient cash flow for the expenditures and increasing gradient cash flow for the income.

For the expenditures: A_{AE} = 1million G = 100000
Present worth of expenditures, $P_E = P_A - P_G$ (decreasing gradient)
$= 1000000.(P/A,15\%,5) - 100000.(P/G,15\%,5)$
$= 1000000.(3.3522) - 100000.(5.7751)$
$= 2774690$

For the income: A_{AI} = 4 million G = 500000
Present worth of income, P_I = 4000000.(P/A,15%,5) + 500000.(P/G,15%,5)
= 16296350

Present worth of net cash flow = PW of income, P_I – PW of expenditures, P_E
= 16296350 – 2774690 = 13521660

11. A company that manufactures auto parts has budgeted $30000 per year to pay for tooling over the next five years. If the company expects to spend $12000 in year 1, how much of a uniform (constant) increase each year is the company expecting in the cost of the tooling if the interest rate is 10% per year? (i.e. what is the arithmetic gradient series equivalent of the uniform-series amounts of $30000?)

In this example, A_T = 30000, A_A = 12000, G = ?

$A_T = A_A + A_G$

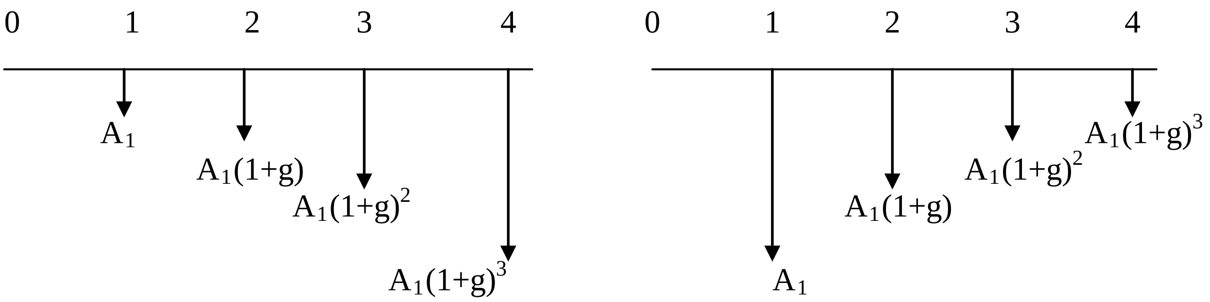
30000 = 12000 + G.(A/G,10%,5) = 12000 + G.(1.8101)

G = $9944 per year.

Geometric Gradient Series

Geometric series of cash flows is the result of cash flow increasing or decreasing from period to period by a constant percentage. In addition to i and n, we use the term,
g = constant rate of change, in decimal form, by which amounts increase or decrease from one period to the next, i.e. for 8% change, g = 0.08.

The cash flow for geometric series take the form:



(a) increasing gradient (b) decreasing gradient

For geometric gradient series, the formula is given to determine the present worth, P_g. If other values such as future value (F) or equivalent uniform series (A) are required, they have to be determined from the value of P_g.

Example:

12. An engineer plans for his retirement and starts a saving program. He saves $5000 for the first year and in the following years he increases his savings by 4% each year. How much will he have in his account after 15 years if the interest earned is 12% per year?

A_1 = 5000, i = 0.12 g = 0.04

Present value, $P_g = 5000.[1 - \{(1+0.04)^{15}/(1+0.12)^{15}\}]/(0.12 - 0.04)$

= 41936

To find the future amount at year 15:

F = 41936.(F/P,12%,15) = $229541 in the account.

Interpolation in Interest Tables

When the factor value for the given i or n is not in the interest tables, the desired values can be determined by either using the formulas or by interpolation. The method of interpolation is demonstrated in the example below.

Examples:

13. Determine the value of the A/P factor for an interest rate of 7.3% per year and n = 10 years.

(A/P,7%,10) = 0.14238
(A/P,8%,10) = 0.14903

Then, (7.3 – 7)/(8 – 7) = x/(0.14903 – 0.14238)
x = 0.00199

and the factor (A/P,7.3%,10) = (A/P,7%,10) + x = 0.14238 + 0.00199 = 0.14437

14. Determine (P/A,8.4%,5) = ?

(P/A,8%,5) = 3.9927
(P/A,9%,5) = 3.8897

Then,
(8.4 – 8)/(9 – 8) = x/(3.8897 – 3.9927)
x = -0.0412

and (P/A,8.4%,5) = (P/A,8%,5) + x = 3.9927 + (-0.0412) = 3.9515