Chapter 3

Probability

Sample space: The set of all possible outcomes of an experiment is called the sample space.

Example:

- In one gender survey: $sample\ space: S = [M, F]$
- Selecting one person: $sample\ space: S = [M, F]$
- Flipping a coin: $sample\ space: S = [H, T]$
- Rolling a dice: $sample\ space: S = [1, 2, 3, 4, 5, 6]$
- Rolling two dice:

$sample\ space$:

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

Event: any set of outcomes of a sample space is called an event

Example: After rolling two dice, we are ınterested to reach sum of the two dice is equal to 5, find the members of this event?

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

$A = \text{sum of the two dice is equal to } 5 = \{(1,4), (2,3), (3,2), (4,1)\}$

$\boldsymbol{A \cup B}$**:** is called the **union** of events A and B to consist of all outcomes that are in A or in B or in both A and B.

$\boldsymbol{A \cap B}$: is called **intersection** of events A and B to consist of all outcomes that are both in A and B.

Null event: the event without any outcomes. (∅)

Exclusive (Disjoint) events: If the intersection of A and B is null event then we say that A and B are disjoint or mutually exclusive.

Complement of an event like A will occur when A does not occur or the set of outcomes that do not A.

Complement of A = A^c

Example: in an experiment we roll a dice. We are ınterested to know about:

S= Sample space- A=even numbers- B=odd numbers- C=bigger than 3- D=smaller than or equal to 5

Find members of every event and also $A \cap B$, $B \cup C$, $A \cup (B \cap C)$, $(A \cup B)^C$, $(B \cap C)^C$

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{ 2, 4, 6\}$

$B = \{1, 3, 5\}$

$C = \{4, 5, 6\}$

$D = \{1, 2, 3, 4, 5\}$

$A \cap B = \{\emptyset\}$

$B \cup C = \{1, 3, 4, 5, 6\}$

$$A \cup (B \cap C) = \begin{cases} B \cap C = \{5\} \\ A \cup (B \cap C) = \{2, 4, 5, 6\} \end{cases}$$

$$(A \cup B)^C = \begin{cases} A \cup B = \{1, 2, 3, 4, 5, 6\} \\ (A \cup B)^C = \{\emptyset\} \end{cases}$$

$$(B \cap C)^C = \begin{cases} B \cap C = \{5\} \\ (B \cap C)^C = \{1, 2, 3, 4, 6\} \end{cases}$$

Example: A fair dice is rolled once and a fair coin is flipped once.

- Find members of event that either the dice will land on 3 and that the coin will land on heads?
- Find members of event that either the dice will land on 3 or that the coin will land on heads?

$S = \{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)\}$

$A = \text{dice will land on 3 and the coin will land on heads} = \{(3, H)\}$

$$B = \text{dice will land on 3 or the coin will land on heads} \\ = \{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (3, T)\}$$

Probability: The word probability is commonly used term that relates to the chances that a particular event will occur when some experiments are performed.

An experiment is any process that produces an observation or outcome,

$$Probability = \frac{Number\ of\ Favorite\ Outcomes}{Number\ of\ Possible\ Outcomes}$$

Properties of probability

1- $0 \le P(A) \le 1 \quad \forall A$
2- $P(S) = 1$, where s is sample space
3- If intersection of A and B is empty ($A \cap B = \emptyset$)

$A \cup A^C = S \quad \rightarrow P(A \cup A^C) = 1$

$P(A^C) = 1 - P(A)$

$P(\emptyset) = 0$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \forall A, B$

$P(A \cup B) = P(A) + P(B) \quad A, B\ are\ disjoint$

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$
$$= \sum_{i=1}^{n} P(A_i) \quad A_1, A_2, \dots, A_n\ are\ disjoint$$

Example: If A and B are disjoint events and $P(A) = 0.2$ & $P(B) = 0.5$

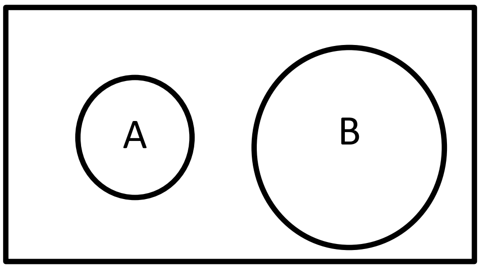
Find: $P(A^c) \quad P(A \cup B) \quad P(A \cap B) \quad P(A^c \cap B)$

$P(A^c) = 1 - P(A) = 1 - 0.2 = 0.8$

$P(A \cup B) = P(A) + P(B) = 0.2 + 0.5 = 0.7$

$P(A \cap B) = 0$

$P(A^c \cap B) = P(B) = 0.5$



Example: Ali has 40% chance of receiving A grade in statistics, 50% chance of receiving A in physics and 86% chance of receiving A in eıther statistics and physics.

Find the probability that he

- Does not receive A in statistics
- Receive A in statistics or physics

$A = \text{Receiving A grade in statistics} \rightarrow \text{P(A)} = 0.4$

$B = \text{Receiving A in physics} \rightarrow \text{P(B)} = 0.5$

$P(A \cap B) = 0.86$

$P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.86 = 0.04$

Example: suppose two dice are rolled. Find the probability that the sum of dices is 6

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

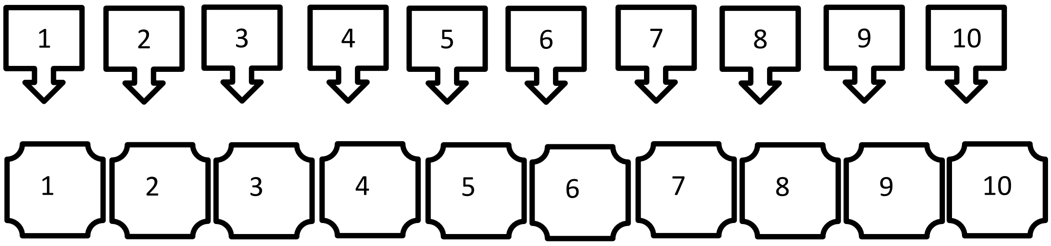
$$A = \text{sum of dices is 6} = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

$$Probability = \frac{Number\ of\ Favorite\ Outcomes}{Number\ of\ Possible\ Outcomes}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

Example: A man has 10 keys to open 10 doors. Find the probability that

- The first key opens a chosen door
- All of 10 keys are tried



$A = \text{first key opens a chosen door}$

$P(A) = \frac{n(A)}{n(S)} = \frac{1}{10}$

$B = \text{All of 10 keys are tried}$

$P(B) = \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8} \times \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{10}$

Conditional probability:

We are often interested in determining probabilities when some partial information concerning the outcomes of the experiment is available. In such situations the probabilities are called conditional probabilities.

Assume that A and B are two events, $P\left(A/_{B}\right) = Probability\ of\ A\ given\ B = \frac{P(A \cap B)}{P(B)}$

Example: consider rolling two dice

- What is the probability that there will be at least one 6 in outcomes?
- We know that the summation is 8, now what is the probability that we have at least one 6?
- We know that the difference is 2, now what is the probability that we have exact one 2?

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

A = at least one 6 in outcomes

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$$P(A) = \frac{n(A)}{n(S)} = \frac{11}{36}$$

B = summation is 8 $= \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

C = exact one 2 $= \{(1,2), (2,1)\ (2,3), (2,4), (2,5), (2,6), (3,2), (4,2), (5,2), (6,2)\}$

$A \cap B = \{(2,6), (6,2)\}$

$$P\left({}^{A}\!/_{B}\right) = Probability\ of\ A\ given\ B = \frac{P(A\cap B)}{P(B)} = \frac{n(A\cap B}{}\!/_{n(S)}}{n(B)\!/_{n(S)}} = \frac{2/36}{5/36} = \frac{2}{5}$$

D = difference is 2 $= \{(1,3), (2,4), (3,5), (4,6), (6,4), (5,3), (4,2), (3,1)\}$

$C \cap D = \{(2,4), (4,2)\}$

$$P\left({}^{C}\!/_{D}\right) = Probability\ of\ C\ given\ D = \frac{P(C \cap D)}{P(D)} = \frac{n(C \cap D)\!/_{n(S)}}{n(D)\!/_{n(S)}} = \frac{2/36}{8/36} = \frac{2}{8} = \frac{1}{4}$$