MATH103

Mathematics for Business and Economics – I

Applications of the Linear Functions and Applications of the Linear Inequalities

Procedure for Solving Word Problems

1. Read the problem carefully and introduce a variable to represent an unknown quantity in the problem.
2. Identify other quantities in the problem (known or unknown) and express unknown quantities in terms of the variable you introduced in the first step.
3. Write a verbal statement using the conditions stated in the problem and then write an equivalent mathematical statement (equation or inequality.)
4. Solve the equation or inequality and answer the questions posed in the problem.
5. Check that the solution solves the original problem.

APPLICATIONS OF EQUATIONS

EXAMPLE 1 Mixture

A chemist must prepare 350 ml *of a chemical solution made up of two parts alcohol and three parts acid. How much of each should be used?*

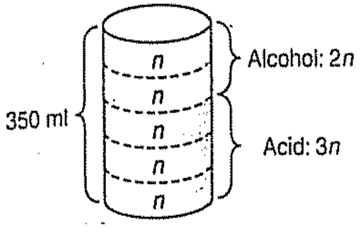


FIGURE 1.1 Chemical solution (Example 1).

Solution: Let n be the number of milliliters in each part. Figure 1.1 shows the situation. From the diagram, we have

$$2n + 3n = 350$$

$$5n = 350$$

$$n = \frac{350}{5} = 70$$

The amount of alcohol is $2n = 2(70) = 140$, and the amount of acid is $3n = 3(70) = 210$.

Some Business Terms:

Fixed cost (or *overhead*) is the sum of all costs that are independent of the level of production, such as rent, insurance, and so on.

Variable cost is the sum of all costs that are dependent on the level of output, such as labor and material.

Total cost is the sum of variable cost and fixed cost:

$$\text{total cost} = \text{variable cost} + \text{fixed cost}$$

Total revenue is the money that the manufacturer receives for selling the output:

$$\text{total revenue} = (\text{price per unit})\,(\text{number of units sold})$$

Profit is total revenue minus total cost:

$$\text{profit} = \text{total revenue} - \text{total cost}$$

EXAMPLE 1 PROFIT

- Anderson Company produces a product for which the variable cost per unit is \$6 and fixed cost is \$80,000. Each unit has a selling price of \$10. How many units must be sold for the company to earn a profit of \$60,000?

Solution: Let q be the number of units that must be sold. (In many business problems, q represents quantity.) Then the variable cost (in dollars) is $6q$. The *total* cost for the business is therefore $6q + 80{,}000$. The total revenue from the sale of q units is $10q$. Since

$$\text{profit} = \text{total revenue} - \text{total cost}$$

our model for this problem is

$$60{,}000 = 10q - (6q + 80{,}000)$$

Solving gives

$$60{,}000 = 10q - 6q - 80{,}000$$

$$140{,}000 = 4q$$

$$35{,}000 = q$$

Thus, 35,000 units must be sold to earn a profit of \$60,000.

EXAMPLE 2 Analyzing Investments

A total of $10,000 was invested in two business ventures, A and B. At the end of the first year, A and B yielded returns of 6% and $5\frac{3}{4}$%, respectively, on the original investments. How was the original amount allocated if the total amount earned was $588.75?

Solution: Let x be the amount (in dollars) invested at 6%. Then $10{,}000 - x$ was invested at $5\frac{3}{4}$%. The interest earned from A was $(0.06)(x)$, and from B it was $(0.0575)(10{,}000 - x)$, which total 588.75. Hence,

$$\begin{aligned}(0.06)x + (0.0575)(10{,}000 - x) &= 588.75\\ 0.06x + 575 - 0.0575x &= 588.75\\ 0.0025x &= 13.75\\ x &= 5500\end{aligned}$$

Thus, $5500 was invested at 6%, and $10,000 − $5500 = $4500 was invested at $5\frac{3}{4}$%.

EXAMPLE 3 Analyzing Investments

Tyrick invests \$15,000, some in stocks and the rest in bonds. If he invests twice as much in stocks as he does in bonds, how much does he invest in each?

Solution

Step 2 Let x = the amount invested in stocks. The rest of the \$15,000 investment (\$15,000 – x) is invested in bonds. We have one more important piece of information to use:

Amount invested in stocks, x = Twice the amount invested in bonds, $15{,}000 - x$

EXAMPLE 3 Analyzing Investments

Step 3 $x = 2(15,000 - x)$ Replace the verbal description with algebraic expressions.

Step 4 $x = 30,000 - 2x$ Distributive property

$3x = 30,000$ Add $2x$ to both sides.

$x = 10,000$ Divide both sides by 3.

Step 5 Tyrick invests $10,000 in stocks and $15,000 – $10,000 = $5000 in bonds.

Step 6 Tyrick's total investment is $10,000 + $5000 = $15,000, and $10,000 (stocks) is twice $5,000 (bonds).

EXAMPLE 4 Solving Problems Involving Simple Interest

Ms. Sharpy invests a total of \$10,000 in blue-chip and technology stocks. At the end of a year, the blue-chips returned 12% and the technology stocks returned 8% on the original investments. How much was invested in each type of stock if the total interest earned was \$1060?

Solution

Step 1 We are asked to find two amounts: that invested in blue-chip stocks and that invested in technology stocks.

If we know how much was invested in blue-chip stocks, then we know that the rest of the \$10,000 was invested in technology stocks.

Step 2 Let x = amount invested in blue-chip stocks. Then $10{,}000 - x$ = amount invested in technology stocks.

Invest	*P*	*r*	*t*	*I* = *Prt*
Blue	*x*	0.12	1	0.12*x*
Tech	10,000 – *x*	0.08	1	0.08(10,000 – *x*)

Interest from Blue-chip + Interest from technology = Total Interest

Step 3 $$0.12x + 0.08(10,000 - x) = 1060$$

$$12x + 8(10,000 - x) = 106,000$$

Step 4 $$12x + 80,000 - 8x = 106,000$$

$$4x = 26,000$$

\$ in blue-chip stocks $x = 6500$

$$10,000 - x = 10,000 - 6500$$

\$ in technology stocks $= 3500$

Step 5 Ms. Sharpy invests $3500 in technology stocks and $6500 in blue- chip stocks.

Step 6

$$\$3500 + \$6500 = \$10,000$$

$$12\% \text{ of } \$6500 = \$780$$

$$8\% \text{ of } \$3500 = \$280$$

$$\text{Total interest earned} = \$1060.$$

EXAMPLE 5: A fence is to enclose a rectangular area of 800 square feet. The length of the plot is twice the width. How much fencing must be used?

Solution: *Let w = width of the plot then 2w = length of the plot*

$$2w\,(w) = 800$$

$$2w^2 = 800$$

$$w^2 = 400$$

$$w = 20$$

So the width of the plot is 20 ft and the length of the plot is 40ft.

So 20+20+40+40 = 120 ft of fencing must be used.

EXAMPLE 6: A builder makes concrete by mixing 1 part clay, 3 parts sand and 5 parts crushed stone. If 765 cubic feet of concrete are needed, how much of each ingredient is needed?

Solution: *Let x = amount of clay, 3x = amount of sand and 5x = amount of crushed stone.*

$$x + 3x + 5x = 765$$

$$9x = 765$$

$$x = 85$$

So 85 cubic feet of clay, 255 cubic feet of sand and 425 cubic feet of stone.

EXAMPLE 7: **A rectangular plot 4 meters by 8 meters is to be used for a garden. The owner wants a border so that 12 square meters of the plot is left for flowers. How wide should the border be?**

Solution: Let X = the width of the border then 4-2x = the width of the plot and 8-2x = the length of the plot

$$(4-2x)(8-2x)=12$$
$$32-16x-8x+4x^2=12$$
$$4x^2-20x+20=0$$
$$4(x-1)(x-5)=0$$
$$x=1$$

So the border should be 1m wide

EXAMPLE 8:**A company wants to know the total sales units required to earn a profit of $100,000. The unit selling price is $20, the variable cost per unit is $15, and the total fixed costs are $600,000. Find the sales units required.**
Solution: Let q = the number of units then the cost = 15q+600000 and revenue = 20q

$$100000=20q-15q-600000$$
$$700000=5q$$
$$140000=q$$

So 140,000 units would need to be sold.

EXAMPLES 9: **Some workers went on strike for 46 days. Before this they earned $7.50 per hour and worked 260 eight-hour days a year. What percent increase is needed in yearly income to make up for the lost time in 1 year?**
Solution: *Let x=the percent increase then x/100 is the decimal form of the percentage*

$$46(8)7.50 = 260(8)7.50\left(\frac{x}{100}\right)$$
$$2760 = 15600\left(\frac{x}{100}\right)$$
$$2760 = 156x$$
$$17.69 = x$$

So about 17.7% increase will be needed.

EXAMPLES 10: ***$20000 is invested in two enterprises so that the total income per year will be $1440.One enterprise pays 6% annually the other 7%. How much was invested into each enterprise?***
Solution: Let x = amount invested in 6% enterprise and 20000-x = amount in 7% enterprise

$$.06x + .075(20000 - x) = 1440$$
$$6x + 150000 - 7.5x = 144000$$
$$-1.5x = -6000$$
$$x = 4000$$

So $4000 was invested at 6% and $16000 was invested at 7%

EXAMPLES 11: ***The price of a product is p dollars each. Suppose a manufacturer will supply $3p^2$-4p units of product to the market and consumers will demand 24-p^2 units. Find the value of p for which the supply will equal the demand.***

Solution:

$$3p^2 - 4p = 24 - p^2$$

$$4p^2 - 4p - 24 = 0$$

$$4(p+2)(p-3) = 0$$

$$p = 3$$

So the price is $3 per product unit.

EXAMPLES 12: ***A circular ventilation duct with diameter 140 mm is attached to a square duct system. To ensure smooth air flow the areas of the circle and square sections must be equal. Find the length of the square side of the section to the nearest millimeter.***

Solution: Let x = the length of the side of the square then $A_s = x^2$ and $A_s = A_c$

$$A_c = \pi(70)^2 \qquad x^2 = 4900\pi$$

$$A_c = 4900\pi \qquad x = 70\sqrt{\pi}$$

$$A_c \approx 154 \qquad x \approx 124$$

So the length of the side of the square section is 124mm.

EXAMPLE 1 Finding a Demand Equation

Suppose the demand per week for a product is 100 units when the price is \$58 per unit and 200 units at \$51 each. Determine the demand equation, assuming that it is linear.

Solution:

Strategy: Since the demand equation is linear. the demand curve must be a straight line. We are given that quantity q and price p are linearly related such that $p = 58$ when $q = 100$ and $p = 51$ when $q = 200$. Thus, the given data can be represented in a q,p-coordinate plane (see Figure 3.15(a)) by points (100, 58) and (200, 51). With these points, we can find an equation of the line—that is, the demand equation.

The slope of the line passing through (100, 58) and (200, 51) is

$$m = \frac{51 - 58}{200 - 100} = -\frac{7}{100}$$

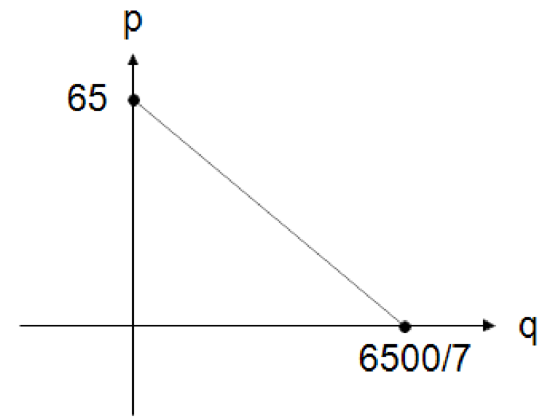
An equation of the line (point-slope form) is

$$p - p_1 = m(q - q_1)$$

$$p - 58 = -\frac{7}{100}(q - 100)$$

Simplifying gives the demand equation

$$p = -\frac{7}{100}q + 65$$



Linear Functions

A *linear function* was described in Section 2.2. Here is another way to do so.

Definition

A function f is a *linear function* if and only if $f(x)$ can be written in the form $f(x) = ax + b$, where a and b are constants and $a \neq 0$.

Suppose that $f(x) = ax + b$ is a linear function, and let $y = f(x)$. Then $y = ax + b$, which is an equation of a straight line with slope a and y-intercept b. Thus, **the graph of a linear function is a straight line that is neither vertical nor horizontal.** We say that the function $f(x) = ax + b$ has slope a.

Note : The graph of a linear function can be drawn by using the same procedures as drawing a line.

EXAMPLE 2 Diet for Hens

In testing an experimental diet for hens, it was determined that the average live weight w (in grams) of a hen was statistically a linear function of the number of days d after the diet began, where $0 \le d \le 50$. Suppose the average weight of a hen beginning the diet was 40 grams and 25 days later it was 675 grams.

a. *Determine w as a linear function of d.*

Solution: Since w is a linear function of d, its graph is a straight line. When $d = 0$ (the beginning of the diet), $w = 40$. Thus, (0, 40) lies on the graph. (See Figure 3.18.) Similarly, (25, 675) lies on the graph. If we set $(d_1, w_1) = (0, 40)$ and $(d_2, w_2) = (25, 675)$, the slope of the line is

$$m = \frac{w_2 - w_1}{d_2 - d_1} = \frac{675 - 40}{25 - 0} = \frac{635}{25} = \frac{127}{5}$$

Using a point–slope form, we have $w - w_1 = m(d - d_1)$

$$w - 40 = \frac{127}{5}(d - 0)$$

$$w - 40 = \frac{127}{5}d$$

$$w = \frac{127}{5}d + 40$$

which expresses w as a linear function of d.

b. *Find the average weight of a hen when* $d = 10$.

Solution: When $d = 10$, $w = \frac{127}{5}(10) + 40 = 254 + 40 = 294$. Thus, the average weight of a hen 10 days after the beginning of the diet is 294 grams.

EXAMPLE 3

A business copier repair company charges a fixed amount plus an hourly rate for service. If a customer is billed $159 for a one-hour service call and $287 for a three hour service call, find the linear function that describes the price of a service call when x is the number of hours of service

Solution

We need to find the equation for a line of the form; y=mx+b

We know two points on the line: (1,159) and (3,287)

So, m = (287 – 159)/(3 – 1) = 64

Use point-slope approach to find the line

- $(y - y_1) = 64(x - x_1)$
- $y = 287 - 64(3) + 65x$
- $y = 95 + 64x$

EXAMPLE 4 Finding a Demand Equation

Example : Suppose consumer will demand 40 units of a Product when the price is $12 per unit and 25 units when The price is $18 each. Find the demand equation Assuming that is linear. Find the price per unit when 30 units are demanded

Solution:

The Points are: (x_1, y_1) = (40,12) , and (x_2, y_2) =(25,18)

Slope Formula: x represent q and y represent p

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{18-12}{25-40} = -\frac{2}{5}$$

Hence an equation of the line is

$$p - 12 = -\frac{2}{5}(q-40)$$

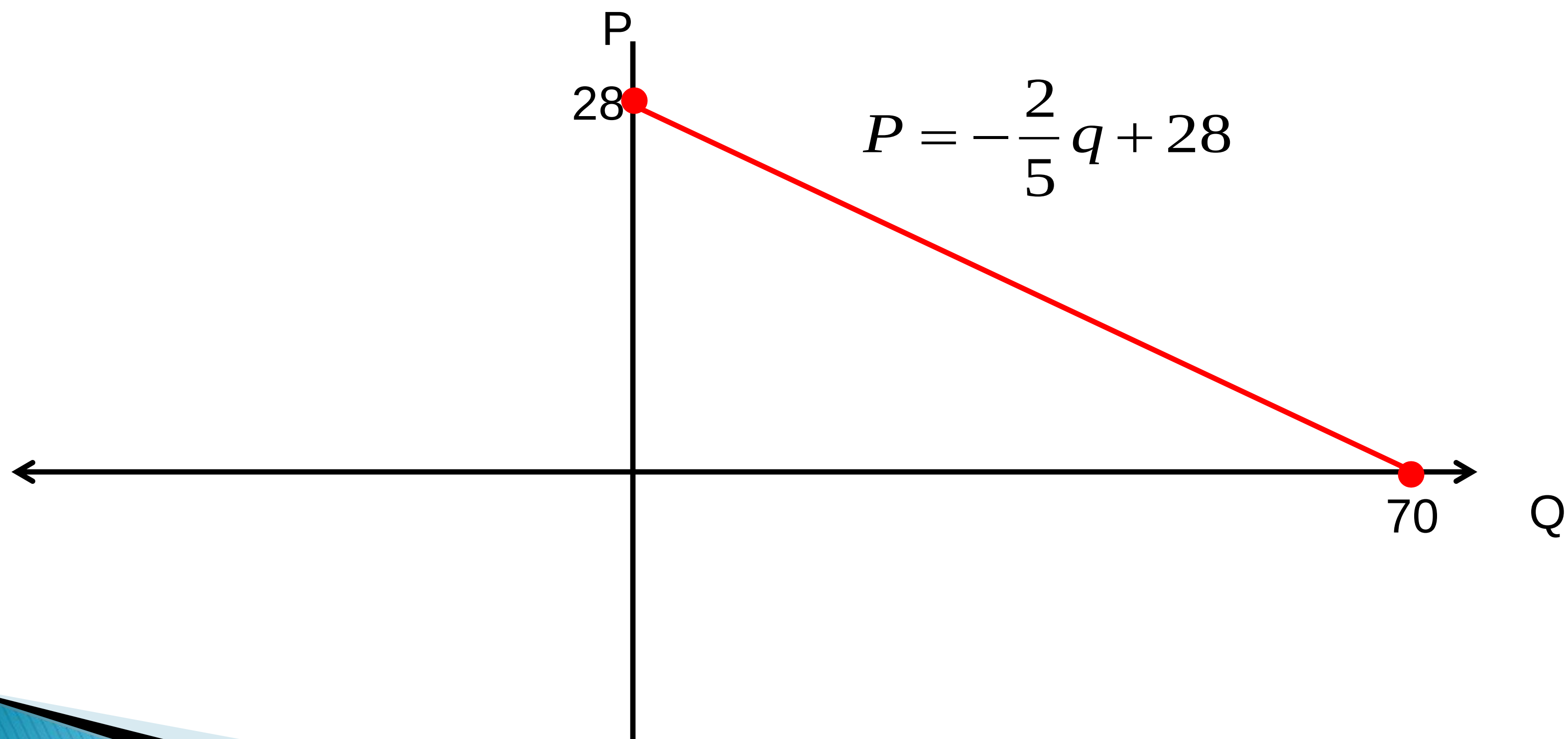
$$p = -\frac{2q}{5} + 28$$

When q=30 ➔ $P = -\frac{2}{5}.30 + 28 = -12 + 28 = 16$

Graph of the demand equation

p intercept q=0 then p=28,

Q intercept p=0 then q=70



EXAMPLE 5 Finding a Demand Equation

Example. (Demand) A dealer can sell 20 electric shavers per day at \$25 per shaver, but he can sell 30 shavers if he charges \$20 per shaver. Determine the demand equation, assuming it is linear.

Solution. Taking the quantity x demanded as the abscissa (or $x-$coordinate) and the price p per unit as the ordinate (or $y-$coordinate), the two points on the demand curve have coordinates

$$x = 20, p = 25 \text{ and } x = 30, p = 20.$$

Thus the points are $(20, 25)$ and $(30, 20)$.

Since the demand equation is linear, it is given by the equation of the straight line passing through the two points $(20, 25)$ and $(30, 20)$.

The slope of the line joining these two points is

$$m = \frac{20-25}{30-20} = -\frac{5}{10} = -0.5.$$

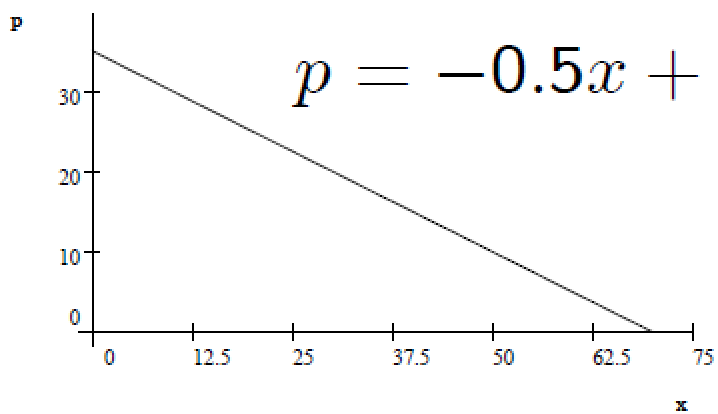
From the point-slope formula, the equation of the line through $(20, 25)$ with slope $m = -\ 0.5$ is

$y - \ y_1 = \ m(x - \ x_1)$. Since $y = p$, we have

$$p - \ 25 = -\ 0.5(x - \ 20)$$

$$\Rightarrow p = -\ 0.5x + 35$$

Which is the required demand equation.



EXAMPLE 6 Finding a Supply Equation

Example Suppose a manufacturer of shoes will place on the market 50 (thousand pairs) when the price is 35 (dollars per pair) and 35 when the price is 30. Find the supply equation, assuming that price p and quantity q are linearly related

Solution:The Points are:(x_1, y_1)=(50,35), and(x_2, y_2)=(35,30)

Slope Formula: x represent q and y represent p

$$m = \frac{p_2 - p_1}{q_2 - q_1} = \frac{30-35}{35-50} = \frac{1}{3}$$

Hence the equation of the line is

$$p - 35 = \frac{1}{3}(q - 50) = \frac{1}{3}q + \frac{55}{3}$$

EXAMPLE 7

Finding a Supply Equation

Example When the price per unit is \$10, the supply will be 80 units per day whereas it will be 90 units at a unit price of \$10.50.
Determine the supply equation, assuming it is linear.
Draw the supply curve.

Solution. Taking the quantity x demanded as the abscissa (or $x-coordinate$) and the price p per unit as the ordinate (or $y-coordinate$), the two points on the supply curve have coordinates

$x = 80, p = 10$ and $x = 90, p = 10.5$.

Thus the points are $(80, 10)$ and $(90, 10.5)$.

Since the supply equation is linear, it is given by the equation of the straight line passing through the two points $(80, 10)$ and $(90, 10.5)$.

The slope of the line joining these two points is

$$m = \frac{10.5 - 10}{90 - 80} = \frac{0.5}{10} = \frac{1}{20}.$$

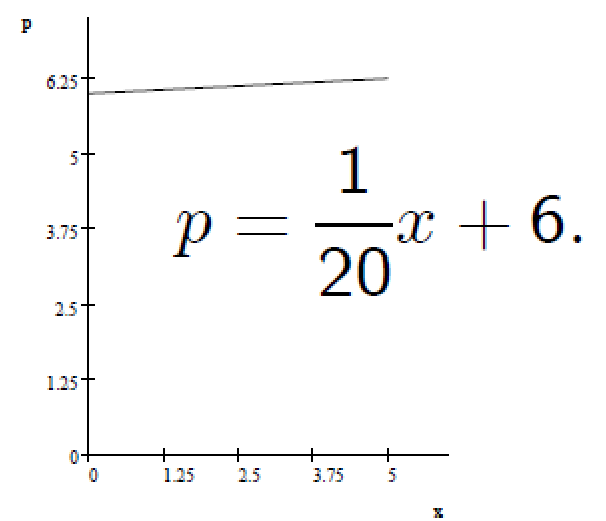
From the point-slope formula, the equation of the line through $(80, 10)$ with slope $m = -\dfrac{1}{20}$ is

$y - \; y_1 = \; m(x - \; x_1)$. Since $y = p$, we have

$$p - \; 10 = \frac{1}{20}(x - 80)$$

$$\Rightarrow p = \frac{1}{20}x + 6.$$

Which is the required supply equation.



EXAMPLE 8 Finding a Cost Equation

Example: Suppose the cost to produce 10 units of a product is $40 and the cost of 20 units is $ 70. If cost c is linearly related to output q, find a linear equation relating c and q. Find the cost to produce 35 units.

Solution:The line passing through (10,40) and (20,70) has slope

$$m=\frac{70-40}{20-10}=3$$

So an equation for the line is: $c-40=3(q-10)$

$$=3q+10$$

If q=35 then c=3(35)+10=115

EXAMPLE 9 Finding an Equation

Example:(Management) Assume that the sales of a certain automobile parts company are approximated by a linear equation. Suppose that sales were \$200000 in 1985 and \$1000000 in 1992.

(a) Find the equation giving the company's yearly sales.

(b) Use this equation to approximate the sales in 1996.

(c) The company wants to negotiate a new contract once sales reach \$2000000.

Solution.

(a) When x =1985, y = 200000, and when x = 1992, y = 1000000. These values give the ordered pairs(1985, 200000) and (7, 1000000).

Find the slope.

$$m = \frac{1000000 - 200000}{1992 - 1985} = \frac{800000}{7}.$$

The $y-intercept$ is 200000, so in the slope-intercept form, $b=200000.$ Therefore, $y=mx+b$

$$y=\frac{800000}{7}x+200000.$$

(b) In 1996, $x=11\ (1996-1985=11)$.

$When\ x=11,$

$$y=\frac{800000}{7}x+200000.$$

$$y=\frac{800000}{7}(11)+200000.$$

$$y=1457000.$$

The estimated sales in 1996 are \$1457000.

$$(c)\text{When } y = 2000000, y = \frac{800000}{7}x + 200000$$

$$200000 = \frac{800000}{7}x + 200000$$

$$1800000 = \frac{800000}{7}x$$

$$12600000 = 800000x$$

$x = 15.75$, $so\ x = 16$. So, $1985 + 16 = 2001$.

Therefore, sales should reach \$2000000 in 2001.

EXAMPLE 10 Finding an Equation

Office equipment was purchased for \$20,000 and will have a scrap value of \$2,000 after 10 years. If its value is depreciated linearly, find the linear equation that relates value (V) in dollars to time (t) in years:

Solution: When $t = 0$, $V = 20{,}000$ and when $t = 10$, $V = 2{,}000$. Thus, we have two ordered pairs (0, 20,000) and (10, 2000). We find the slope of the line using the slope formula.
The y intercept is already known
The slope is (2000-20,000)/(10 – 0) = -1,800.
when $t = 0$, $V = 20{,}000$, so the y intercept is b=20,000
Therefore, by slope-intercept form(y=mx+b)
equation is $V(t) = -1{,}800t + 20{,}000$.

EXAMPLE 11 Depreciation

A new television is $120 per year, and it is worth $340 after 4 years. Find a function that describes the value of this television if x is the age of the television in years.

Solution:

V : The value of the television

$V_{original}$: The original value of the television

x : the age of the television

The value of the television per year is 120, m=120

$$\textit{The} \text{ equation of the value of the television is}$$

$$V = V_{original} - 120x$$

$$340 = V_{original} - 120(4)$$

$$V_{original} = 340 + 480 = 820$$

$$V = -120x + 820$$

EXAMPLE 12 Appreciation

A house purchased for $189000 is expected to double in value in 18 year. Find a linear equation that describes the house’s value after x years.

Solution:

Let x = the number of years after the house is purchased.

x=18, V=378000

$$V_{original} = 189000$$

$$V = 2V_{original} = 2(189000) = 378000$$

$$V = V_{original} + mx$$

$$378000 = 189000 + m(18)$$

$$m = 10500$$

The equation is $V = 10500x + 189000$

Applications of Inequalities

EXAMPLE 1 Profit

For a company that manufactures aquarium heaters, the combined cost for labor and material is \$21 per heater. Fixed costs (costs incurred in a given period, regardless of output) are \$70.000. If the selling price of a heater is \$35, how many must be sold for the company to earn a profit?

Solution:

Let q be the number of heaters sold. Then their cost is $21q$.

The total cost for the company is therefore $21q + 70,000$.

The total revenue from the sale of q heaters will be $35q$.

$$\text{profit} = \text{total revenue} - \text{total cost}$$

and we want profit > 0. Thus,

$$\begin{aligned} \text{total revenue} - \text{total cost} &> 0 \\ 35q - (21q + 70,000) &> 0 \\ 14q &> 70,000 \\ q &> 5000 \end{aligned}$$

Therefore, at least 5001 heaters must be sold for the company to earn a profit.

EXAMPLE 2: **A company manufactures a product that has a unit selling price of $20 and a unit cost of $15. If fixed costs are $600,000, determine the least number of units that must be sold for the company to have a profit.**
Solution: *Let the x be the number of units then 20x = total revenue and 15x+600000 = total cost*

$$20x - 15x - 600000 > 0$$
$$5x > 600000$$
$$x > 120000$$

So we need at least 120001 units to make a profit

EXAMPLE 3: ***A company invests $30,000 of surplus funds at two annual rates: 5% and 6.75%. The company wishes to have an annual yield of no less than 6.5%.What is the least amount of money the company can invest at 6.75%?***
Solution: *Let x = amount at 6.75% then 30000-x= amount at 5%*

$$.05(30000 - x) + .0675x \geq .065(30000)$$
$$1500 - .05x + .0675x \geq 1950$$
$$.0175x \geq 450$$
$$x \geq 25714.29$$

So need to invest at least $25714.29

EXAMPLE 4: The cost of publication of each copy of a magazine is $0.65. It is sold to dealers for $0.60 each and the amount received for advertising is 10% of the amount received for all magazines issued beyond 10000. What is the least number of magazines that can be published without loss?

Solution: Let x = the number of units then 0.65x = total cost and 0.6x+0.1(0.6)(x-10000) = Total revenue

$$
\begin{aligned}
&.60x + .10(.60)(x-10000) - .65x \geq 0 \\
&.60x + .06x - 600 - .65x \geq 0 \\
&.01x \geq 600 \\
&x \geq 60000
\end{aligned}
$$

So at least 60,000 magazines must be published.

EXAMPLE 5: **At present a manufacturer has 2500 units of a product in stock. This month, the product sells for $4 a unit. Next month the unit price will increase by $0.50. The manufacturer wants the total revenue received from the sale of 2500 units to be no less than $10,750. What is the maximum number of units that can be sold this month?**

Solution:*Let x = the number of units then the total revenue is 4x+4.5(2500-x)*

$$
\begin{aligned}
&4x + 4.5(2500-x) \geq 10750 \\
&4x + 11250 - 4.5x \geq 10750 \\
&-.50x \geq -500 \\
&x \leq 1000
\end{aligned}
$$

So need to sell a max of 1000 units this month.