MATH103

Mathematics for Business and Economics – I

The System of Equations and Applications of the systems

System of Linear Equations

System of two linear equations in two variables can be represented as

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in R$

A solution of a system of equations in two variables *x* and *y* is an ordered pair of numbers (*x*,*y*), all resulting equations in the system are true.

The **solution set of a system of equations** is the set of all solutions of the system.

There are 3 ways of solving system of linear equations

1) Graphical method
2) Substitution method
3) Elimination method

EXAMPLE 1 Verifying a Solution

Verify that the ordered pair (3, 1) is the solution of the system of linear equations

$$\begin{cases} 2x - y = 5 \\ x + 2y = 5 \end{cases}$$

Solution

Replace x by 3 and y by 1.

$$2x - y = 5 \qquad\qquad x + 2y = 5$$

$$2(3) - (1) = 5 \qquad\qquad (3) + 2(1) = 5$$

$$6 - 1 = 5 \checkmark \qquad\qquad 3 + 2 = 5 \checkmark$$

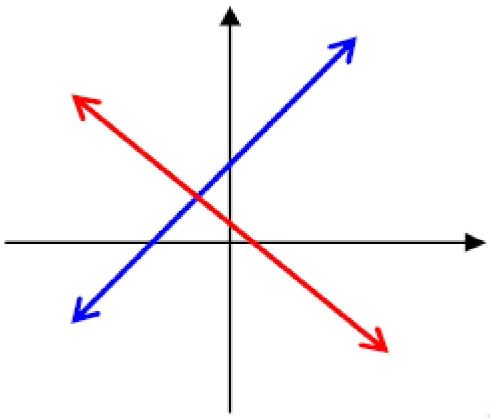
(3, 1) satisfies both equations, so it is the solution.

1) Solving Systems by Graphing

- **Solution of the system of linear equations –** any ordered pair in a system that makes *all* equations true.
- When we graph a system of two linear equations in two variables, one of the following **three outcomes will occur.**

i) Graphs intersect at one point.

The system is ***consistent and has one solution.*** Since neither equation is a multiple of the other, they are independent.



EXAMPLE 1 Solving a System by the Graphical Method

Use the graphical method to solve the system equations

$$\begin{cases} 2x - y = 4 & (1) \\ 2x + 3y = 12 & (2) \end{cases}$$

Solution

Step 1 Graph both equations on the same coordinate axes.

Find the x- and y-intercepts and graph.

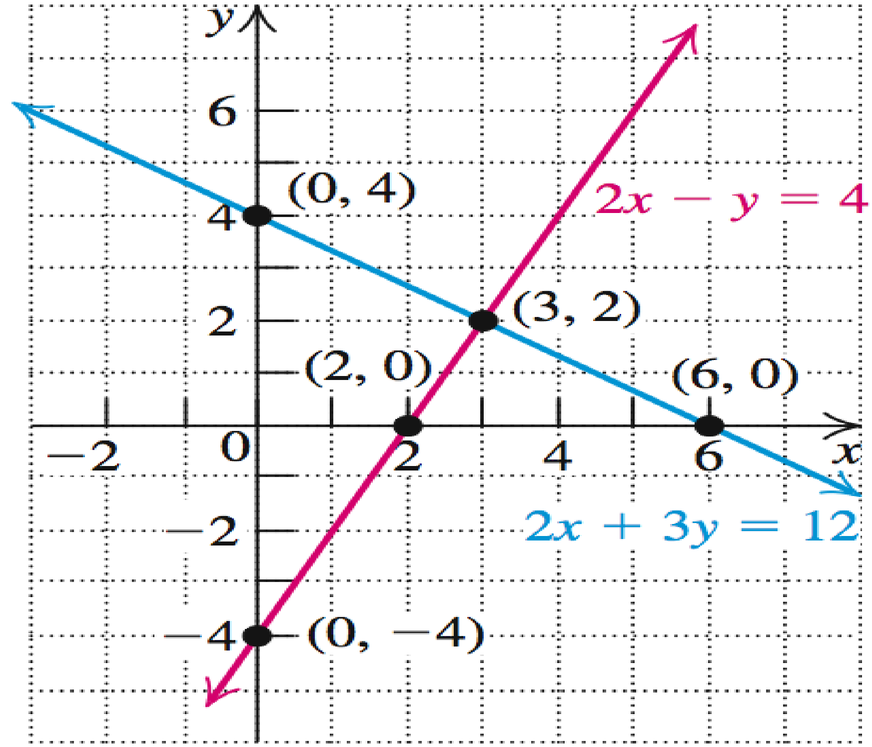
Set $x = 0$ in $2x - y = 4$ and solve for y: $y = -4$

Set $y = 0$ in $2x - y = 4$ and solve for x: $x = 2$

Set $x = 0$ in $2x + 3y = 12$ and solve for y: $y = 4$

Set $y = 0$ in $2x + 3y = 12$ and solve for x: $x = 6$

Step 2 Find the point(s) of intersection of the two graphs.



The point of intersection for the two graphs is (3, 2).

Step 3 Check your solution(s).

Replace *x* by 3 and *y* by 2.

$$2x - y = 4 \qquad 2x + 3y = 12$$

$$2(3) - 2 = 4 \qquad 2(3) + 3(2) = 12$$

$$6 - 2 = 4 \qquad 6 + 6 = 12$$

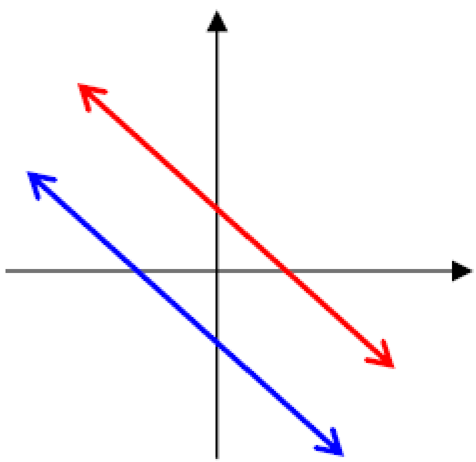
Yes Yes

Step 4 **Write the solution set for the system.**

The solution set is {(3, 2)}.

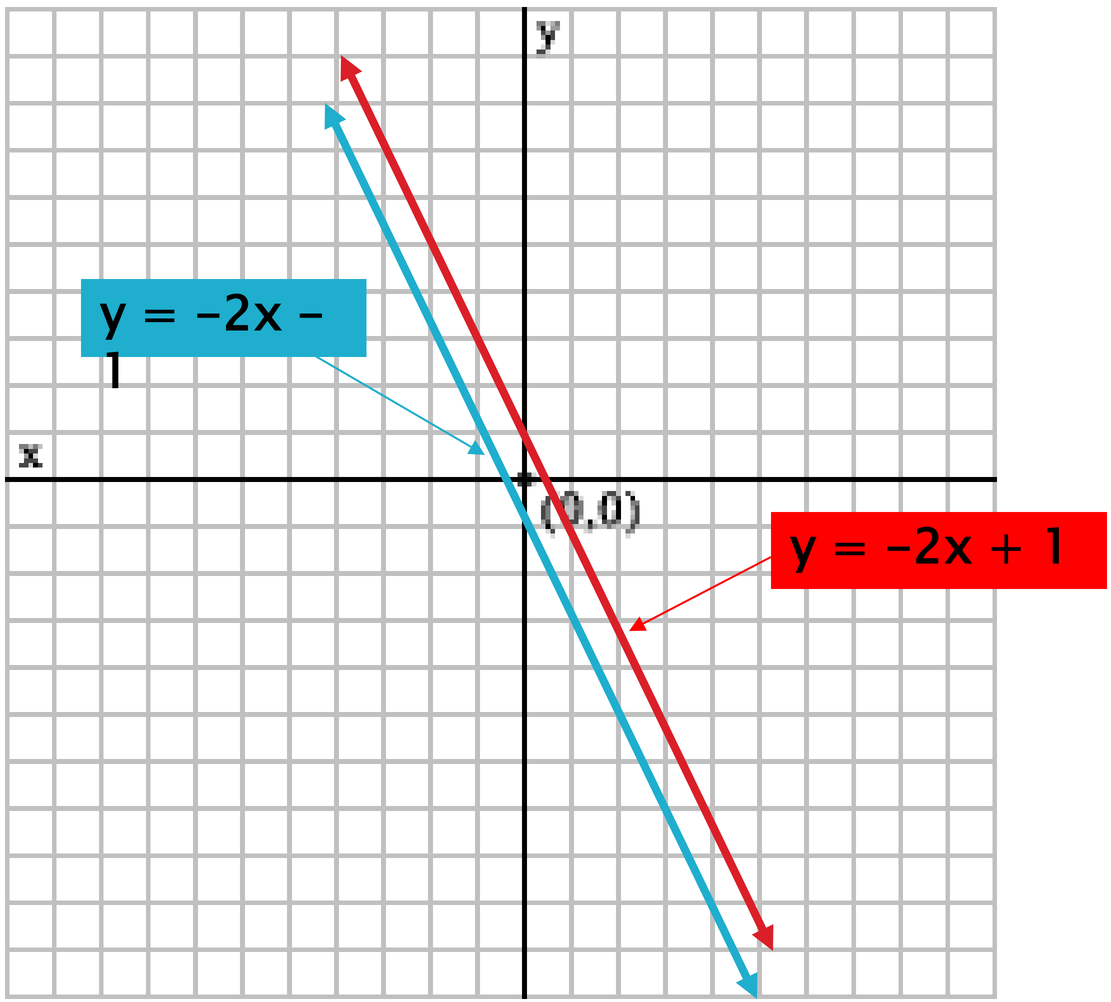
ii) Graphs are parallel.

When two lines are parallel, there are no points of Intersection. The system is *inconsistent because* there is **no solution.** Since the equations are not equivalent, they are independent.



Systems with No Solutions

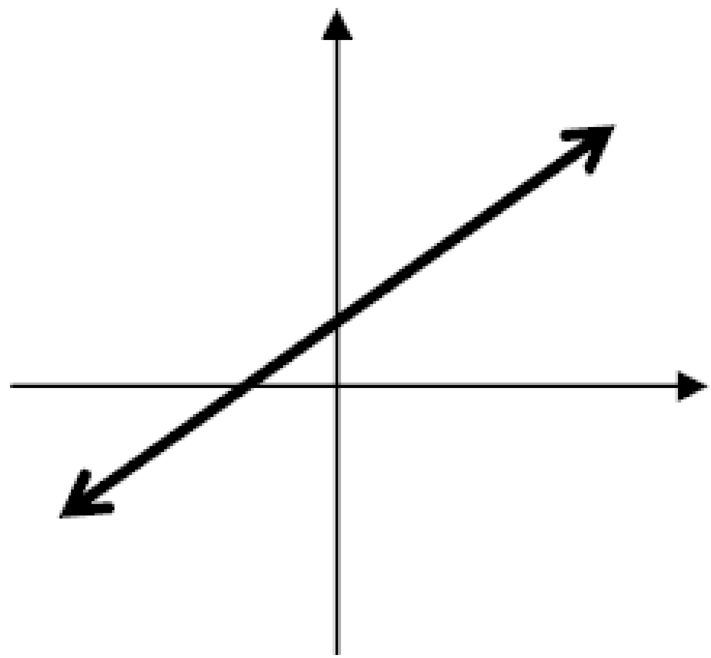
- Solve by graphing. $y = -2x + 1$
  $y = -2x - 1$



The lines are parallel, so there is no solution.

iii) Equations have the same graph.

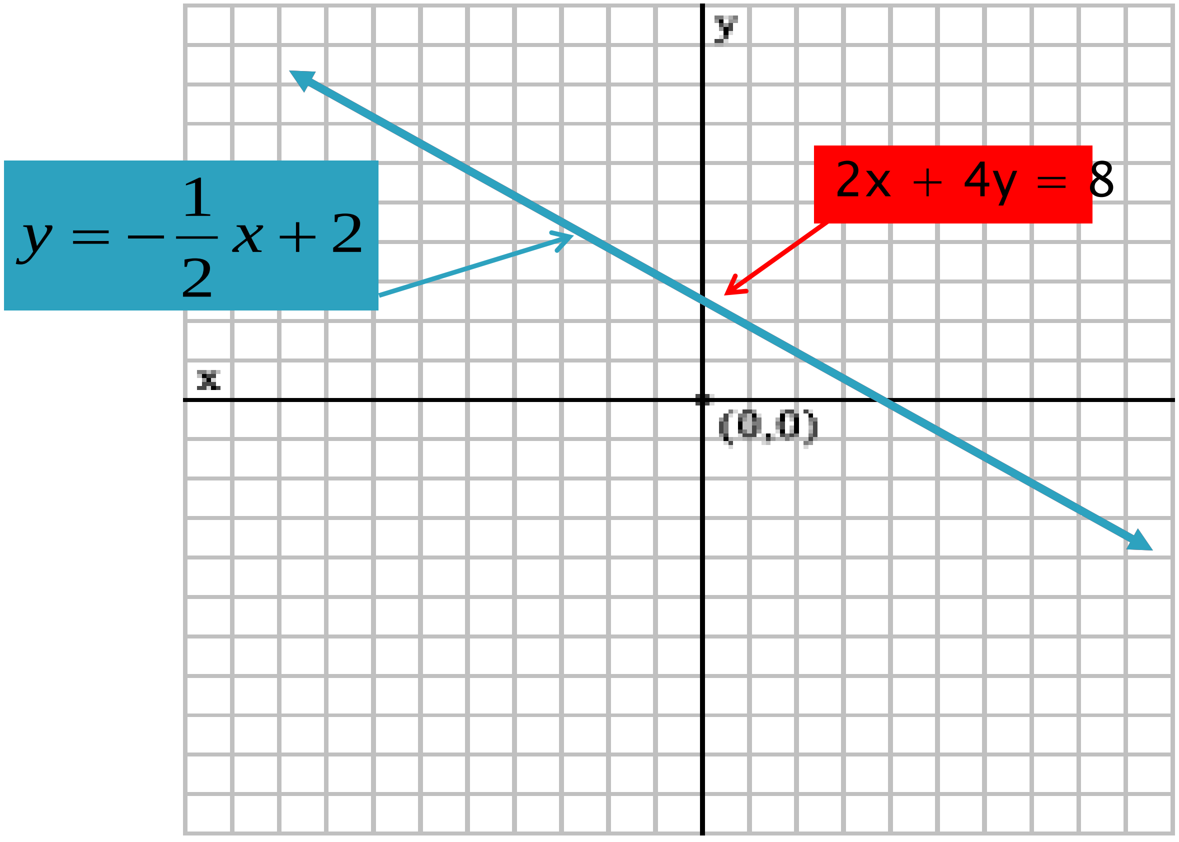
The system is ***consistent and has an infinite* number of solutions. The** equations are dependent since they are equivalent.



Systems with Infinitely Many Solutions

- Solve by graphing. 2x + 4y = 8

$$y = -\frac{1}{2}x + 2$$



The graphs are the same line. There are infinitely many solutions.

Numbers of Solutions of Systems of Linear Equations

- Different slopes – the lines intersect, so there is one solution.
- Same slope, different y-intercepts – the lines are parallel, so there are no solutions.
- Same slope, same y-intercept – the lines are the same so there are infinitely many solutions.

- Solve by graphing. Check your solution.

1. $y = x + 2$
   $y = -2x + 2$

2. $y = 1$
   $y = x$

- Graph each system. Tell whether the system has one solution, no solution, or infinitely many solutions.

3. $y = 2x$
   $y = 2x - 5$

4. $y = -3x + 1$
   $y = 3x + 7$

2) Solving Systems Using Substitution

Another method for solving systems of equations is the **substitution method**.

Step 1. **Solve for one variable**. Choose one of the equations, and express one of the variables in terms of the other variable.

Step 2. **Substitute**. Substitute the expression obtained in Step 1 into the other equation to obtain an equation in one variable.

Step 3. **Solve** the equation obtained in Step 2.

Step 4. **Back-substitute**. Substitute the value(s) you obtained in Step 3 back into the expression you found in Step 1. This gives the solutions.

Step 5. **Check**. Check your answer(s) in the original equation.

Solve by substitution . $3x-2y=-1$

$$x+y=3$$

Step1. Let's solve for *x in the second equation:* $x+y=3$

Subtract *y from both sides* $x=3-y$

Step2. Now substitute $3-y$ in the place of x in the first equation.

$$3(3-y)-2y=-1$$

Step3. Solve this equation for y

Remove parentheses:	$9-3y-2y=-1$
Collect like terms:	$9-5y=-1$
Subtract 9 from both sides	$-5y=-10$
Divide both sides by 5	$y=2$

Step4. Now substitute back into the equation 2 from part 1:

$$x=3-2=1$$

So the solution to the system is $(1,2)$.

Step5. Check your result !!!!

EXAMPLE 2 Solve by substitution

Solve using the substitution method.

$$3y + 2x = 4$$
$$-6x + y = -7$$

Solution:

Step 1–Solve the second equation for y because it has a coefficient of 1.

$$-6x + y = -7$$
$$y = 6x – 7$$

Step 2–Write an equation containing only one variable and solve.

$$3y + 2x = 4$$
$$3(6x – 7) + 2x = 4$$
$$18x – 21 + 2x = 4$$
$$20x – 21 = 4$$
$$20x = 25 \rightarrow x = 1.25$$

Step 3 - Solve for the other variable in either equation.

$$-6(1.25) + y = -7$$
$$-7.5 + y = -7 \rightarrow y = 0.5$$

Since x = 1.25 and y = 0.5, the solution is (1.25 , 0.5).

Solve by substitution $3x+2y=11$

$$y=x+3$$

Solution:

1. They have already solve for *y in the second equation. So say "thank you" and proceed.*

2. Now substitute *x+3 in the place of y in the first equation.*

$$3x+2(x+3)=11$$

3. Solve this equation:

Remove parentheses: $3x+2x+6=11$

Collect like terms: $5x+6=11$

Subtract 6 from both sides $5x=5$

Divide both sides by 5

4. Now substitute back into the equation from part 1:

$$y=1+3=4$$

So the solution to the system is (1,4)

EXAMPLE 4 Solve by substitution

Solve by substitution $x=2y-3$

$$2x-3y=-5$$

Solution:

1. They have already solve for *x in the first equation. So say "thank you" and proceed.*
2. Now substitute $2y-3$ in the place of *x in the second equation.*

$$2(2y-3)-3y=-5$$

3. Solve this equation: $4y-6-3y=-5$

$$y-6=-5$$

$$y=1$$

4. Now substitute back into the equation from part 1:

$$x=2(1)-3=-1$$

So the solution to the system is $(-1,1)$

Solve by substitution $y=2-3x$

$$6x+2y=7$$

Solution:

1. They have already solve for *y in the first equation. So say "thank you" and proceed.*

2. Now substitute $2-3x$ *in the place of y in the second equation.*

$$6x+2(2-3x)=7$$

3. Solve this equation:

$$6x+4-6x=7$$

$$4=7$$

This is called a contradiction.

So there is no solution. The system is *inconsistent.*

EXAMPLE 6 Solve by substitution

Solve by substitution $3x+4y=18$

$$2x-y=1$$

Solution:

1. Let's solve for *y in the second equation:*

$y=2x-1$

2. Now substitute $2x-1$ in the place of *y in the first equation.* $3x+4(2x-1)=18$

3. Solve this equation: $3x+8x-4=18$

$$11x=22$$

$$x=2$$

4. Now substitute back into the equation $y=2x-1$ *from part 1:*

$$y = 3$$

So the solution to the system is (2,3)

Solve by substitution $3x+y=4$

$$9x+3y=12$$

Solution:

1. Let's solve for *y in the first equation:*

$$y=4-3x$$

2. Now substitute $4-3x$ *in the place of y in the second equation.*

$$9x+3(4-3x)=12$$

3. Solve this equation: $9x+12-9x=12$

$$12=12$$

This is always true. So any value of *x will work. So the system has infinitely many solutions! The system is dependent.*

EXAMPLE 8

Attempting to Solve an Inconsistent System of Equations

Solve the system of equations.

$$\begin{cases} x + y = 3 & (1) \\ 2x + 2y = 9 & (2) \end{cases}$$

Solution

Step 1. Solve equation (1) for y in terms of x.

$$x + y = 3$$

$$y = 3 - x$$

Step 2. Substitute into equation (2).

$$2x + 2y = 9$$

$$2x + 2(3 - x) = 9$$

EXAMPLE 8

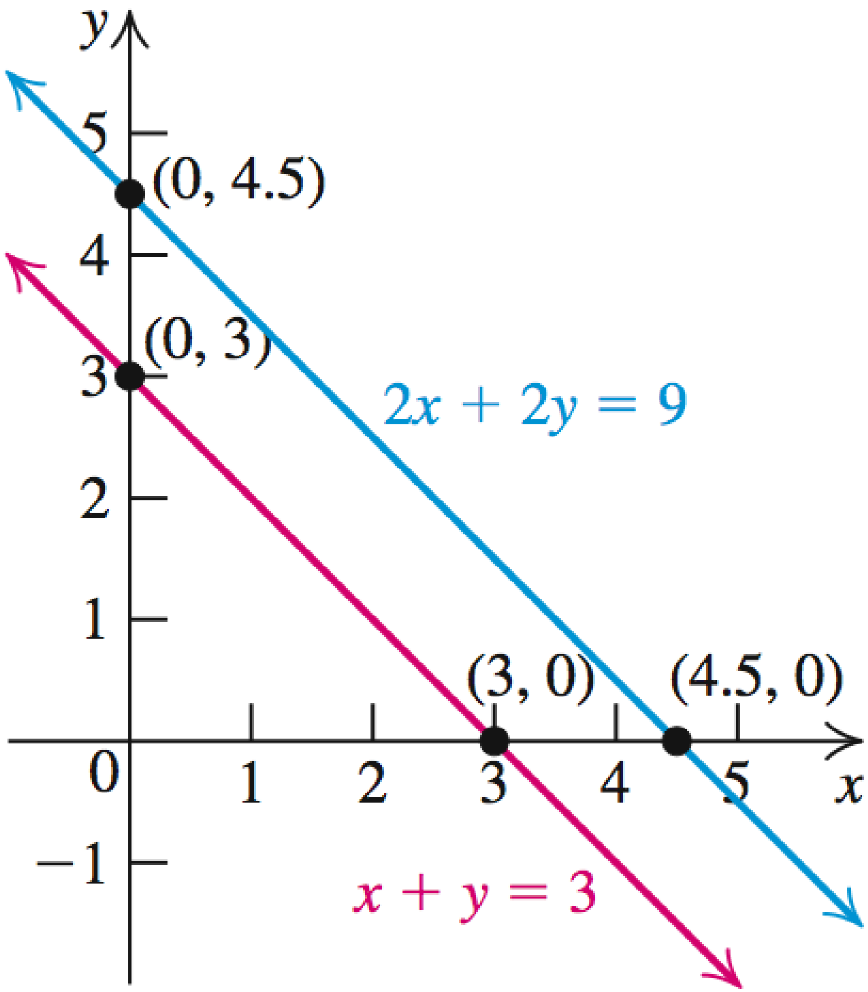
Attempting to Solve an Inconsistent System of Equations

Solution continued

$$2x + 6 - 2x = 9$$

$$0 = 3$$

Since the equation 0 = 3 is false, the system is inconsistent. The lines are parallel, do not intersect and the system has no solution.



3) Solving Systems Using Elimination

You can use the Addition and Subtraction Properties of Equality to solve a system by using the **elimination method.**

Step 1. Decide which variable you wish to eliminate.

Step 2. Multiply each equation by a number that will make the coefficients of the chosen variable the same in value but opposite in sign.

Step 3. Add the two new equations together. This eliminates your chosen variable.

Step 4. Solve for the remaining variable in this new equation.

Step 5. Substitute this value for the variable back into one of the original equations. This is called **back substitution.**

EXAMPLE	Using the Elimination method

Solve the system. $\begin{cases} 3x - 4y = 12 & (1) \\ 2x + 5y = 10 & (2) \end{cases}$

Solution

Step 1 Select the variable *y* for elimination.

$$\begin{array}{r} 15x - 20y = 60 \\ 8x + 20y = 40 \\ \hline 23x \qquad\quad = 100 \end{array}$$

Step 2

Step 3

$$x = \frac{100}{23}$$

Step 4 Back-substitute $x = \frac{100}{23}$ in equation (2).

$$2x + 5y = 10$$

$$2\left(\frac{100}{23}\right) + 5y = 10$$

$$5y = 10 - \frac{200}{23} = \frac{30}{23}$$

$$y = \frac{6}{23}$$

Step 5 The solution set is $\left\{\left(\frac{100}{23}, \frac{6}{23}\right)\right\}$.

Step 6 You can verify that $x = \frac{100}{23}$ and $y = \frac{6}{23}$ satisfy both equations (1) and (2).

EXAMPLE 1 Solve by elimination

Elimination by Addition

- For our system, we will seek to eliminate the *x* variable. The coefficients are 2 and −5. Our goal is to obtain coefficients of *x* that are additive inverses of each other.
- We can accomplish this by multiplying the first equation by 5, and the second equation by 2.
- Next, we can add the two equations to eliminate the *x*–variable.
- Solve for *y*
- Substitute *y* value into original equation and solve

$$\left.\begin{array}{l} 2x-7y=3 \\ -5x+3y=7 \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l} 5(2x-7y)=5(3) \\ 2(-5x+3y)=2(7) \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l} 10x-35y=15 \\ -10x+6y=14 \end{array}\right\} \rightarrow$$

$$0x-29y=29 \rightarrow$$

$$y=-1$$

$$2x-7(-1)=3 \rightarrow$$

$$2x+7=3 \rightarrow$$

$$2x=-4$$

$$x=-2$$

$$(-2,-1)$$

EXAMPLE 2 Solve by elimination

Solve by elimination. $5x - 6y = -32$

$3x + 6y = 48$

Step 1-2-3 – Eliminate y because the sum of the coefficients of y Is zero.

$$\begin{array}{r} 5x - 6y = -32 \\ \underline{3x + 6y = 48} \\ 8x + 0 = 16 \end{array}$$

step4 $8x = 16 \rightarrow x = 2$

Step5 – Solve for the eliminated variable y using either of the original equations. $3x + 6y = 48$

$$3(2) + 6y = 48$$

$$6 + 6y = 48$$

$$6y = 42 \rightarrow y = 7.$$

Since x = 2 and y = 7, the solution is (2 , 7).

CHECK $5(2) - 6(7) = -32$

$$10 - 42 = -32$$

$$-32 = -32$$

Solve by the elimination. $2x + 5y = -22$

$10x + 3y = 22$

Solution: Step1–Eliminate one variable. Start with the given system.

$2x + 5y = -22$

$10x + 3y = 22$

Step2- To prepare for eliminating x, multiply the first equation by 5.

$5\,(2x + 5y = -22)$

Step3-Subtract the equations to eliminate x.

$$\begin{aligned} 10x + 25y &= -110 \\ \underline{10x + 3y} &\underline{= \quad 22} \\ 0 + 22y &= -132 \end{aligned}$$

Step 4 – Solve for y. $0 + 22y = -132$

$22y = -132 \Rightarrow y = -6$

Step 5 – Solve for the eliminated variable using either of the original equations. $2x + 5y = -22$

$2x + 5(-6) = -22$

$2x – 30 = -22$

$2x = 8 \Rightarrow x = 4$ **The solution is (4 , -6).**

EXAMPLE 4 Solve by elimination

Solve by the elimination $4x + 2y = 14$

$7x - 3y = -8$

Solution:Step 1 – Eliminate one variable. Start with the given system.

$$4x + 2y = 14$$
$$7x - 3y = -8$$

Step2-To prepare for eliminating y, multiply the first equation by 3 and the other equation by 2. $3(4x + 2y = 14)$

$2(7x - 3y = -8)$

Step3-Add the equations to eliminate y.

$$12x + 6y = 42$$
$$\underline{14x - 6y = -16}$$
$$26x + 0 = 26$$

Step 4 – Solve for x. $26x = 26$ ➔ $x = 1$

Step 5 – Solve for the eliminated variable y using either of the original equations.

$$4x + 2y = 14$$
$$4(1) + 2y = 14$$
$$4 + 2y = 14$$
$$2y = 10 \Rightarrow y = 5$$

The solution is (1 , 5).

EXAMPLE 5 — Solve by elimination

Solve the system by the Elimination method

$$3x - 2y = -1$$
$$x + y = 3$$

Solution:

1. Let's eliminate y.
2. Multiply equation 2 by 2. Leave equation 1 as it is:

$$3x - 2y = -1$$
$$2x + 2y = 6$$

3. Add the two equations together: $5x = 5$
4. Solve for x *in this equation:* $x = 1$
5. Back substitute in the second equation.

$$1 + y = 3$$
$$y = 2$$

Solution is (1,2)

Check the solution:

First equation: $3.1 - 2.2 = -1$

Second equation: $2.1 + 2.2 = 6$

EXAMPLE 6

Solve by elimination

Solve the system by the Elimination method

$$x-2y=4$$
$$3x+4y=2$$

1. Let's eliminate y.
2. Multiply equation 1 by 2

$$2x-4y=8$$
$$3x+4y=2$$

3. Add the two equations together:

$$5x=10$$

4. Solve for x *in this equation:*

$$x=2$$

5. Back substitute in the first equation.

$$2-2y=4$$
$$-2y=2$$
$$y=-1$$

The solution is $(2,-1)$

Solve by the elimination method:

$$x+3y=2$$
$$3x+9y=6$$

Solution:

1. Let's eliminate y
2. Multiply equation the first equation by -3

$$-3x-9y=-6$$
$$3x+9y=6$$

3. Add the two equations together:

$$0=0$$

This is always true so the system is dependent. There are infinitely many solutions.

EXAMPLE 7

Solve by elimination

Solve by the elimination method:

$$2x+4y=3$$
$$3x+6y=8$$

1. Let's eliminate *x*.
2. Multiply equation 1 by −3 and multiply equation 2 by 2

$$-6x-12y=-9$$
$$6x+12y=16$$

3. Add the two equations together:

$$0=7$$

This is a contradiction. The system is inconsistent. There are no solutions

EXAMPLE 9 Finding the Equilibrium Point application

Find the equilibrium point if the supply and demand functions for a new brand of digital video recorder (DVR) are given by the system

$$p = 60 + 0.0012x \quad (1)$$

$$p = 80 - 0.0008x \quad (2)$$

where *p* is the price in dollars and *x* is the number of units.

Solution

Substitute the value of *p* from equation (1) into equation (2) and solve the resulting equation.

$$p = 80 - 0.0008x$$

$$60 + 0.0012x = 80 - 0.0008x$$

$$0.0012x = 20 - 0.0008x$$

$$0.002x = 20$$

$$x = \frac{20}{0.002}$$

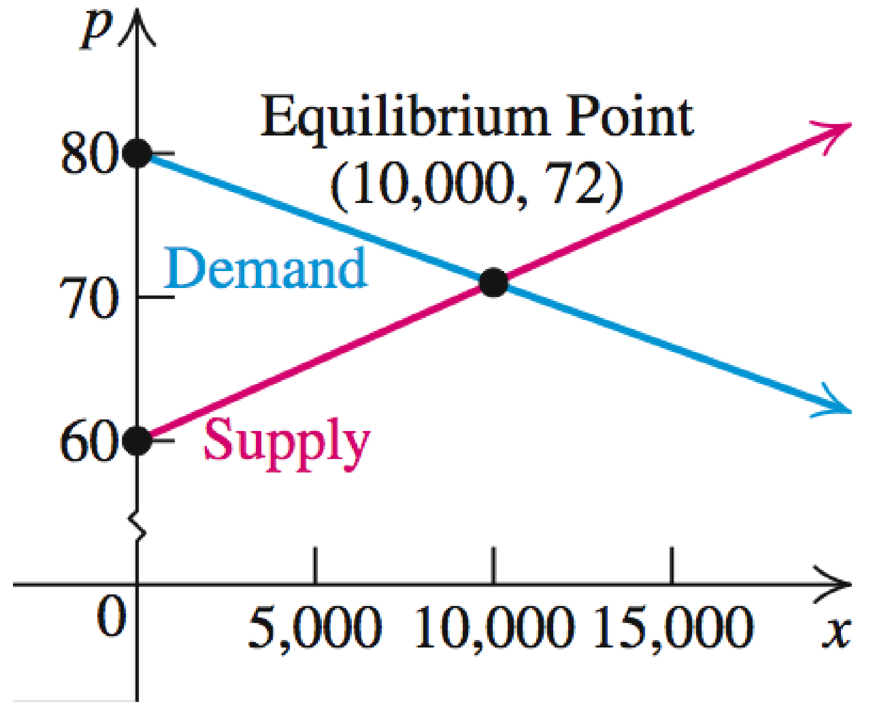
$$x = 10,000$$

To find the price *p* back-substitute

X = 10,000.

$$
\begin{aligned}
p &= 60 + 0.0012x \\
&= 60 + 0.0012(10,000) \\
&= 72
\end{aligned}
$$

The equilibrium point is (10,000, 72).
You can verify that this point satisfies both equations.



EXAMPLE 10 Finding the Equilibrium Point application

Find the equilibrium point if the supply and demand functions are given by the system

$$p = \frac{3}{100}q + 2 \qquad (1)$$

$$p = -\frac{7}{100}q + 12 \quad (2)$$

where *p* is the price in dollars and *q* is the number of units.

Solution

Substitute the value of *p* from equation (1) into equation (2) and solve the resulting equation.

$$\frac{3}{100}q + 2 = -\frac{7}{100}q + 12$$

$$\frac{10}{100}q = 10$$

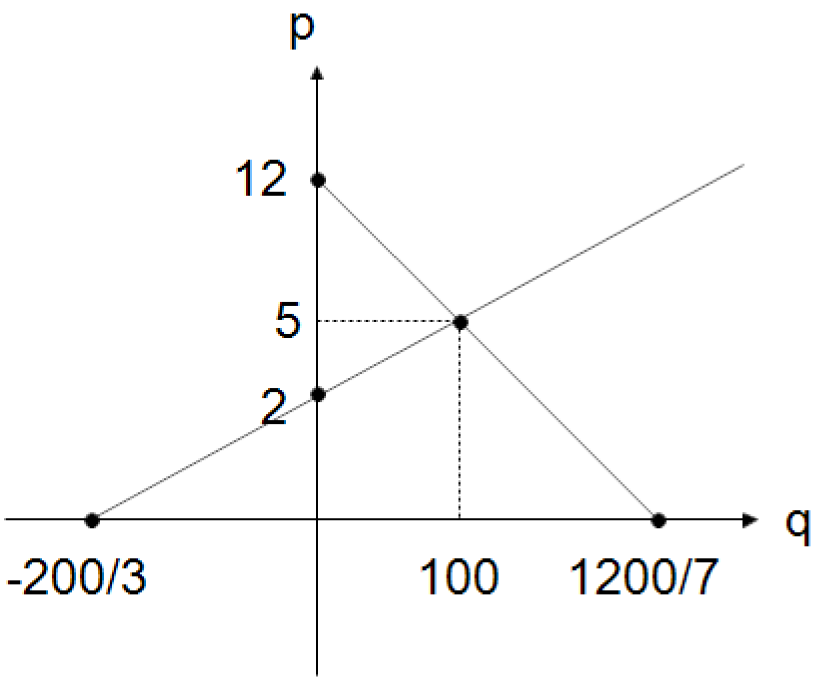
$$q = 100$$

To find the price *p* back-substitute

q = 100.

$$p = \frac{3}{100}(100) + 2 = 5$$

The equilibrium point is (100, 5).
You can verify that this point satisfies both equations.



Systems of Linear Equations in Three Variables

Definitions

A **linear equation in the variables** $\boldsymbol{x_1, x_2, \ldots, x_n}$ is an equation that can be written in the form.

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b.$$

where *b* and the coefficients $a_1, a_2, \ldots, a_n$, are real numbers. The subscript *n* may be any positive integer.

A **system of linear equations** (or a **linear system**) in three variables is a collection of two or more linear equations involving the same variables.

An ordered triple (a, b, c) is a solution of a system of three equations in three variables x, y, and z if each equation in the system is a true statement when *a*, *b*, and *c* are substituted for x, y, and z respectively.

EXAMPLE 1 Solving Three – Variable Linear System

Solve

$$\begin{cases} 2x + y + z = 3 & (30) \\ -x + 2y + 2z = 1 & (31) \\ x - y - 3z = -6 & (32) \end{cases}$$

Solution: This system consists of three linear equations in three variables. From Equation (32), $x = y + 3z - 6$. By substituting for x in Equations (30) and (31), we obtain

$$\begin{cases} 2(y + 3z - 6) + y + z = 3 \\ -(y + 3z - 6) + 2y + 2z = 1 \\ x = y + 3z - 6 \end{cases}$$

Simplifying gives

$$\begin{cases} 3y + 7z = 15 & (33) \\ y - z = -5 & (34) \\ x = y + 3z - 6 \end{cases}$$

Note that x does not appear in Equations (33) and (34). Since any solution of the original system must satisfy Equations (33) and (34), we shall consider their solution first:

$$\begin{cases} 3y + 7z = 15 & (33) \\ y - z = -5 & (34) \end{cases}$$

From Equation (34), $y = z - 5$. This means that we can replace Equation (33) by

$$3(z-5) + 7z = 15 \qquad \text{that is} \qquad z = 3$$

Since z is 3, we can replace Equation (34) with $y = -2$. Hence, the previous system is equivalent to $\begin{cases} z = 3 \\ y = -2 \end{cases}$

The original system becomes $\begin{cases} z = 3 \\ y = -2 \\ x = y + 3z - 6 \end{cases}$

from which $x = 1$. The solution is $x = 1$, $y = -2$, and $z = 3$, which you should verify.