Math104 Midterm Exam, Spring 2006-07

Name Surname: Number: Group: 09/04/07 90 min

Q1. Find the matrix A if $\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$. (Hint: $(B^{-1})^{-1} = B$) (15 pnts)

Solution:

$$\left(\left(A^T - 2I\right)^{-1}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1}, \qquad \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$A^T - 2I = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad A^T - 2I + 2I = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} + 2I, \quad A^T = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

$\left(A^T\right)^T = A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}^T$. So the matrix A is $\boxed{A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}}$

Q2. Solve the following system by **inverse matrix method**. (20 pnts)

$$x + 5y = 1$$
$$3x + 9y = 2$$

Solution:

Inverse matrix method is $X = A^{-1}B$.

The matrix form of this system is Ax=B, $\begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The matrix A is $A = \begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix}$, $|A| = \begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = 9 - 15 = -6$

The inverse off the matrix A is $A^{-1} = -\frac{1}{6}\begin{bmatrix} 9 & -5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -3/2 & 5/6 \\ 1/2 & -1/6 \end{bmatrix}$.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3/2 & 5/6 \\ 1/2 & -1/6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/2 + 5/6 \\ 1/2 - 1/6 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$. So the solution set is $S = \left\{(x, y) : \left(\frac{1}{6}, \frac{1}{6}\right)\right\}$

Q3. Given the function $y = f(x) = \frac{3}{2}x^4 - 4x^3 + 17$, (15 pnts)

a) Find the equation of the tangent line touching to this curve at x=1.

Solution:

$y' = f'(x) = 6x^3 - 12x^2$, the slope is $m = y' = f'(1) = 6 - 12 = -6$.

When $x = 1$, then $y = \frac{3}{2} - 4 + 17 = \frac{29}{2}$.

The tangent line is $y - y_1 = m(x - x_1)$

$$y - \frac{29}{2} = -6(x-1), \quad y = -6x + \frac{41}{2}.$$

b) Find all x on f(x) where the tangent line to the curve becomes parallel to the x-axis.

Solution:

The slope of the line which is parallel to the x-axis (horizontal line) is zero.

$y' = f'(x) = 6x^3 - 12x^2 = 6x^2(x-2) = 0$

The values of x are $x = 0$ and $x = 2$

Q4. Given the function $y = f(x) = \frac{3}{2}x^4 - 4x^3 + 17$, (20 pnts)

a) Determine the increasing and decreasing regions on f(x)

Solution:

$y' = f'(x) = 6x^3 - 12x^2 = 6x^2(x-2) = 0$. The critical points are $x = 0$ and $x = 2$.

x	0	2	
$6x^2$	+	+	+
$x-2$	-	-	+
y'	-	-	+
y	decreasing	decreasing	increasing

So, f is increasing on $(2, \infty)$ and

f is decreasing on $(-\infty, 0) \cup (0, 2)$.

b) Determine the concave up and concave down regions on f(x)

Solution:

$y'' = f''(x) = 18x^2 - 24x = 6x(3x-4) = 0$. The roots of f are $x = 0$ and $x = \frac{4}{3}$ are inflection points which is shown on the table.

x	0	4/3	
$6x$	-	+	+
$3x-4$	-	-	+
y''	+	-	+
y	Concave up	Concave down	Concave up

So, f is concave up on $(-\infty, 0) \cup \left(\frac{4}{3}, \infty\right)$ and f is concave down on $\left(0, \frac{4}{3}\right)$.

Q5. Given the function $y = f(x) = \frac{3}{2}x^4 - 4x^3 + 17$, (20 pnts)

a) Locate and identify the critical points(min/max) on f(x)

Solution:

$y' = f'(x) = 6x^3 - 12x^2 = 6x^2(x-2) = 0$, the critical points are $x = 0$ and $x = 2$.

When $x = 0$ then $f(0) = 17$ and when $x = 2$ then $f(2) = \frac{3}{2}(16) - 4(8) + 17 = 9$.

From the first sign table, the the relative minimum point is $\boxed{(2,9)}$.

b) Locate all inflection points on f(x)

Solution:

When $x = \frac{4}{3}$ then $f\left(\frac{4}{3}\right) = \frac{3}{2}\left(\frac{4}{3}\right)^4 - 4\left(\frac{4}{3}\right)^3 + 17 = \frac{331}{27}$

From the second sign table, the inflection points are $\boxed{(0,17)}$ and $\boxed{\left(\frac{4}{3}, \frac{331}{27}\right)}$

Q6. Find the derivative of $y = f(x) = \sqrt{2x^3+x}\,\ln(x^3+2x+5) + e^{3x^2+2x+1} + \frac{x^e}{x+1} + e^5$ (20 pnts)

Solution:

$$y' = f'(x) = \frac{1}{2}\left(2x^3+x\right)^{-\frac{1}{2}}\left(6x^2+1\right)\ln\left(x^3+2x+5\right) + \frac{3x^2+2}{x^3+2x+5}\sqrt{2x^3+x} + \left(6x+2\right)e^{3x^2+2x+1} + \frac{ex^{e-1}\left(x+1\right)-x^e}{\left(x+1\right)^2} + 0$$

The simple form of this derivative is

$$y' = f'(x) = \frac{1}{2}\frac{\left(6x^2+1\right)}{\sqrt{2x^3+x}}\ln\left(x^3+2x+5\right) + \frac{3x^2+2}{x^3+2x+5}\sqrt{2x^3+x} + \left(6x+2\right)e^{3x^2+2x+1} + \frac{\left(e-1\right)x^e + ex^{e-1}}{\left(x+1\right)^2}$$