Chapter 3

Analysis of experimental Data

Errors will creep into all experiments regardless of the care which is exerted.

Some of these errors are of a random nature, and some will be due to gross blunders on the part of the experimenter.

If such bad points fall outside the range of normal expected random deviations, they may discard on the basis of some consistent statistical data analysis.

Consistent: The elimination of data points must be consistent and should not be dependent on what ought to be.

Causes and types of experimental errors

The data may be single sample or multisample.

Ex. For single sample

If one measures pressure with a pressure gage and a single instrument is the only one used for the entire set of observations, then some of the error that is present in the measurement will be sampled only once no matter how many time the reading is repeated.

Ex. For multi sample

If more than one pressure gage is used for the same total set of observations, then we might say that a multi sample experiment has been performed. The number of observations will then determine the success of this multi sample experiment in accordance with accepted statistical principles.

Experimental uncertainty: The possible value the error may have.

Experimental errors: The real errors in experimental data are those factors that are always not clear to some extent and carry some amount of uncertainty.

Types of Errors

1. Errors gross blunders in apparatus or instrument construction which may invalidate the data. Careful experimenter will be able to eliminate most of these errors.
2. Certain fixed errors which will cause repeated readings to be in error by roughly the same amount but for some unknown reason (systematic errors, or bias errors).
3. Random errors: usually follow a certain statistical distribution, but not always. In many instances it is very difficult to distinguish between fixed errors and random errors.

The experimentalist may sometimes use theoretical methods to estimate the magnitude of a fixed error.

Ex. Consider the measurement of the temperature of a hot gas stream flowing in a duct with a mercury in glass thermometer.

he heat may be conducted from the stem of the thermometer, out of the body and into the surroundings. Heat transfer from the gas to the stem of the thermometer, and consequently the temperature of the stem must be lower than that of the hot gas. Therefore the temperature we read on the thermometer is not the true temperature of the gas and it will not make any difference how many readings are taken we shall always have an error resulting from the heat transfer condition of the stem of the thermometer. This is fixed error, and its magnitude may be estimated with theoretical calculations based on known thermal properties of the gas and the glass thermometer.

Error analysis on a commonsense basis.

The uncertainity of final results is due to the instrument accuracy and competence of the people using the instruments. This may be done by common sense analysis of the data which may take many forms.

1. One rule of thumb that could be used is that the error in the result is equal to the maximum error in any parameter used to calculate the result.
2. Another common sense analysis would combine all the errors in the most detrimental way in order to determine the maximum error in the final result.

Ex. Calculation of electric power

$P=I\ E$

Where I and E are measured as

$E = 100\ V \pm 2V$

$I = 10\ A \pm 0.2\ A$

The normal value of power is

$P= 100 \times 10 = 1000\ W$

The worst possible in variations in the voltage and current are,

$P_{max} =(100 + 2) \times (10 + 0.2) = 1040.4\ W$ (uncertainty = $\frac{1040.4-1000}{1000}$=0.0404)

$P_{min} = (100 – 2) \times (10 – 0.2) = 960.4\ W$

The uncertainty in the power is +4.04%, -3.96 %

Uncertainty Analysis

A more precise method of estimating uncertainty in experimental results has been presented by Kline and Mc Clintock. The method is based on a careful specification of the uncertainties in the various primary experimental measurements.

Ex. A certain pressure reading might be expressed

P = 100 kPa ± 1 kPa

or P = 100 kPa ± 1 kPa (20 to 1)

In other words, the experimenter is willing to bet with 20 to 1 odds that the pressure measurement is within ± 1 kPa.

Suppose a set of measurements is made and the uncertainty in each measurement may be expressed wiyh the same odds.

$R = R (x_1, x_2, x_3, ,\ldots\ldots\ldots\ldots x_n)$

Where R is a given function of the independent variables $x_1, \ldots . x_n$.

Let w_R : the uncertainty in the result

$w_1 , w_2 \ldots . w_n$: the uncertanities in the independent variables.

If the uncertainties in the independent variables are all given wiyh the same odds. Then

$$W_R = \left[\left(\frac{\partial R}{\partial X_1}W_1\right)^2 + \left(\frac{\partial R}{\partial X_2}W_2\right)^2 + \cdots . + \left(\frac{\partial R}{\partial X_n}W_n\right)^2\right]^{1/2}$$

Uncertainties for product Functions

$R = x_1^{a1}x_2^{a2} \ldots\ldots\ldots . . x_n^{an}$ take the partial differentiations

$$\frac{\partial R}{\partial x_i} = x_1{}^{a_1}\ x_2{}^{a_2}\,(aix_1{}^{a_{i-1}}) \ldots\ldots . . x_n{}^{a_n} \qquad \text{(divide with R)}$$

Therefore $\frac{1}{R}\frac{\partial R}{\partial x_i} = \frac{a_i}{x_i}$

$$\frac{W_R}{R} = \left[\sum\left(\frac{a_i\, w x_i}{x_i}\right)^2\right]^{1/2} \qquad \text{Fractional uncertainty} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ..1$$

Ex. P = I E

I = 10 ± 0.2 A

E = 100 ± 2 V

$$\frac{W_R}{R} = \left[\left(\frac{2}{100}\right)^2 + \left(\frac{0.2}{10}\right)^2\right]^{1/2} = 2.83\%$$

Uncertainty for Additive Functions

$R = R (a_1x_1, a_2x_2, a_3x_3, ,\ldots\ldots\ldots\ldots A_nx_n) = \sum a_i\ x_i$

$$\frac{\partial R}{\partial x_i} = a_i$$

Then the uncertainty in the result

$$W_R = \left\{\sum \left[\left(\frac{\partial R}{\partial x_i}\right)^2 \; W_{x_i}^2\right]\right\}^{1/2}$$

$$W_R = \left\{\sum \left[\left(a_i W_{x_i}\right)^2 \right]\right\}^{1/2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

Note eqn 1 and eqn 2 may be used in combination when the result function involves both product and additive terms.

Ex. The resistance of a copper wire is given by

$$R = R_o[1 + \propto (T - 20)]$$

Where R_o = 6 Ω ±0.3% is the resistance at 20 ^oC

α = 0.004 $^oC^{-1}$

T = 30 ± 1 temperature of wire

Calculate the resistance of the wire and its uncertainty.

$R = 6[1 + 0.004(30 - 20)]$= 6. 24 Ω normal resistance

$\frac{\partial R}{\partial R_o} = 1 + \propto (T - 20)$ = 1+0.004 (30-20) = 1.04 Ω

$\frac{\partial R}{\partial \alpha} = Ro(T - 20)$ = 6 (30 – 20) = 60 Ω

$\frac{\partial R}{\partial T} = Ro \propto$ = 6 (0.004)= 0.024 Ω

W_{Ro} = 6 * 0.003 = 0.018 Ω

W_α = 0.004 * 0.01 = 4 x 10^{-5} $^oC^{-1}$

W_T = 1 ^oC

$W_R = [(1.04x0.018)^2 + (60x0.00004)^2 + (0.024x1)^2]^{1/2}$ = 0.0305 Ω

or 0.0305 Ω / 6.24 Ω = 0.49%

UNCERTAINTY IN POWER MEASUREMENT. The two resistors R and R_s are connected in series as shown in the accompanying figure. The voltage drops across each resistor are measured as

$$E = 10\text{ V} \pm 0.1\text{ V } (1\%)$$
$$E_s = 1.2\text{ V} \pm 0.005\text{ V } (0.467\%)$$

along with a value of

$$R_s = 0.0066\ \Omega \pm 1/4\%$$

From these measurements determine the power dissipated in resistor R and its uncertainty.

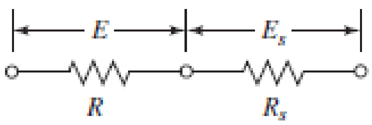

Figure Example 3.2

Solution

The power dissipated in resistor R is

$$P = EI$$

The current through both resistors is $I = E_s/R_s$ so that

$$P = \frac{EE_s}{R_s} \qquad \textbf{[a]}$$

The nominal value of the power is therefore

$$P = (10)(1.2)/(0.0066) = 1818.2\text{ W}$$

The relationship for the power given in Eq. (a) is a product function, so the fractional uncertainty in the power may be determined from Eq. (3.2a). We have

$$a_E = 1 \qquad a_{E_s} = 1 \qquad \text{and} \qquad a_{R_s} = -1$$

so that

$$\frac{w_P}{P} = \left[\left(\frac{a_E w_E}{E}\right)^2 + \left(\frac{a_{E_s} w_{E_s}}{E_s}\right)^2 + \left(\frac{a_{R_s} w_{R_s}}{R_s}\right)^2\right]^{1/2}$$

$$= \left[(1)^2\left(\frac{0.1}{10}\right)^2 + (1)^2\left(\frac{0.005}{1.2}\right)^2 + (-1)^2(0.0025)^2\right]^{1/2} = 0.0111$$

Then

$$w_P = (0.0111)(1818.2) = 20.18\text{ W}$$

SELECTION OF MEASUREMENT METHOD. A resistor has a nominal stated value of 10 Ω ± 1 percent. A voltage is impressed on the resistor, and the power dissipation is to be calculated in two different ways: (1) from $P = E^2/R$ and (2) from $P = EI$. In (1) only a voltage measurement will be made, while both current and voltage will be measured in (2). Calculate the uncertainty in the power determination in each case when the measured values of E and I are

$$E = 100 \text{ V} \pm 1\% \qquad \text{(for both cases)}$$
$$I = 10 \text{ A} \pm 1\%$$

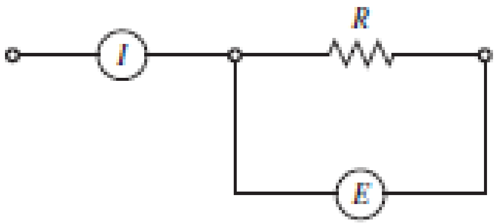

Figure Example 3.3 Power measurement across a resistor.

Solution

The schematic is shown in the accompanying figure. For the first case we have

$$\frac{\partial P}{\partial E} = \frac{2E}{R} \qquad \frac{\partial P}{\partial R} = -\frac{E^2}{R^2}$$

and we apply Eq. (3.2) to give

$$w_P = \left[\left(\frac{2E}{R}\right)^2 w_E^2 + \left(-\frac{E^2}{R^2}\right)^2 w_R^2\right]^{1/2} \qquad \textbf{[a]}$$

Dividing by $P = E^2/R$ gives

$$\frac{w_P}{P} = \left[4\left(\frac{w_E}{E}\right)^2 + \left(\frac{w_R}{R}\right)^2\right]^{1/2} \qquad \textbf{[b]}$$

Inserting the numerical values for uncertainty gives

$$\frac{w_P}{P} = [4(0.01)^2 + (0.01)^2]^{1/2} = 2.236\%$$

For the second case we have

$$\frac{\partial P}{\partial E} = I \qquad \frac{\partial P}{\partial I} = E$$

and after similar algebraic manipulation we obtain

$$\frac{w_P}{P} = \left[\left(\frac{w_E}{E}\right)^2 + \left(\frac{w_I}{I}\right)^2\right]^{1/2} \qquad [c]$$

Inserting the numerical values of uncertainty yields

$$\frac{w_P}{P} = [(0.01)^2 + (0.01)^2]^{1/2} = 1.414\%$$

INSTRUMENT SELECTION. The power measurement in Example 3.2 is to be conducted by measuring voltage and current across the resistor with the circuit shown in the accompanying figure. The voltmeter has an internal resistance R_m, and the value of R is known only approximately. Calculate the nominal value of the power dissipated in R and the uncertainty for the following conditions:

$R = 100\,\Omega$ (not known exactly)

$R_m = 1000\,\Omega \pm 5\%$

$I = 5\text{ A} \pm 1\%$

$E = 500\text{ V} \pm 1\%$

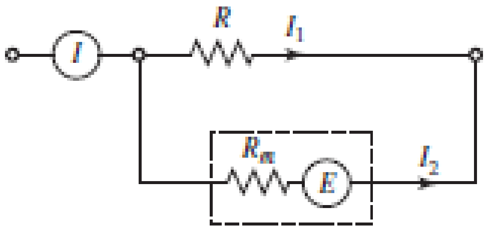

Figure Example 3.4 Effect of meter impedance on measurement.

Solution

A current balance on the circuit yields

$$I_1 + I_2 = I$$

$$\frac{E}{R} + \frac{E}{R_m} = I$$

and

$$I_1 = I - \frac{E}{R_m} \qquad [a]$$

The power dissipated in the resistor is

$$P = EI_1 = EI - \frac{E^2}{R_m} \qquad \textbf{[b]}$$

The nominal value of the power is thus calculated as

$$P = (500)(5) - \frac{500^2}{1000} = 2250 \text{ W}$$

In terms of known quantities the power has the functional form $P = f(E, I, R_m)$, and so we form the derivatives

$$\frac{\partial P}{\partial E} = I - \frac{2E}{R_m} \qquad \frac{\partial P}{\partial I} = E$$

$$\frac{\partial P}{\partial R_m} = \frac{E^2}{R_m^2}$$

The uncertainty for the power is now written as

$$w_P = \left[\left(I - \frac{2E}{R_m}\right)^2 w_E^2 + E^2 w_I^2 + \left(\frac{E^2}{R_m^2}\right)^2 w_{R_m}^2\right]^{1/2} \qquad \textbf{[c]}$$

Inserting the appropriate numerical values gives

$$\begin{aligned} w_P &= \left[\left(5 - \frac{1000}{1000}\right)^2 5^2 + (25 \times 10^4)(25 \times 10^{-4}) + \left(25 \times \frac{10^4}{10^6}\right)^2 (2500)\right]^{1/2} \\ &= [16 + 25 + 6.25]^{1/2}(5) \\ &= 34.4 \text{ W} \end{aligned}$$

or

$$\frac{w_P}{P} = \frac{34.4}{2250} = 1.53\%$$

In order of influence on the final uncertainty in the power we have

1. Uncertainty of current determination
2. Uncertainty of voltage measurement
3. Uncertainty of knowledge of internal resistance of voltmeter

WAYS TO REDUCE UNCERTAINTIES. A certain obstruction-type flowmeter (orifice, venturi, nozzle), shown in the accompanying figure, is used to measure the flow of air at low velocities. The relation describing the flow rate is

$$\dot{m} = CA\left[\frac{2g_c p_1}{RT_1}(p_1 - p_2)\right]^{1/2} \quad \textbf{[a]}$$

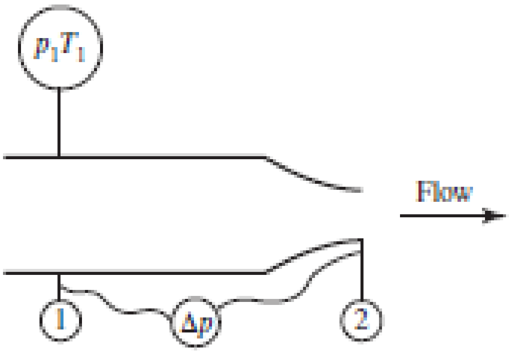

Figure Example 3.5 Uncertainty in a flowmeter.

where C = empirical-discharge coefficient
A = flow area
p_1 and p_2 = upstream and downstream pressures, respectively
T_1 = upstream temperature
R = gas constant for air

Calculate the percent uncertainty in the mass flow rate for the following conditions:

$C = 0.92 \pm 0.005$ (from calibration data)
$p_1 = 25$ psia ± 0.5 psia
$T_1 = 70°F \pm 2°F$ $\quad T_1 = 530°R$
$\Delta p = p_1 - p_2 = 1.4$ psia ± 0.005 psia (measured directly)
$A = 1.0\ \text{in}^2 \pm 0.001\ \text{in}^2$

Solution

In this example the flow rate is a function of several variables, each subject to an uncertainty.

$$\dot{m} = f(C, A, p_1, \Delta p, T_1) \quad \textbf{[b]}$$

Thus, we form the derivatives

$$\frac{\partial \dot{m}}{\partial C} = A\left(\frac{2g_c p_1}{RT_1}\Delta p\right)^{1/2}$$

$$\frac{\partial \dot{m}}{\partial A} = C\left(\frac{2g_c p_1}{RT_1}\Delta p\right)^{1/2}$$

$$\frac{\partial \dot{m}}{\partial p_1} = 0.5CA\left(\frac{2g_c}{RT_1}\Delta p\right)^{1/2} p_1^{-1/2} \quad \textbf{[c]}$$

$$\frac{\partial \dot{m}}{\partial \Delta p} = 0.5CA\left(\frac{2g_c p_1}{RT_1}\right)^{1/2} \Delta p^{-1/2}$$

$$\frac{\partial \dot{m}}{\partial T_1} = -0.5CA\left(\frac{2g_c p_1}{R}\Delta p\right)^{1/2} T_1^{-3/2}$$

The uncertainty in the mass flow rate may now be calculated by assembling these derivatives in accordance with Eq. (3.2). Designating this assembly as Eq. (*c*) and then dividing by Eq. (*a*) gives

$$\frac{w_{\dot{m}}}{\dot{m}} = \left[\left(\frac{w_C}{C}\right)^2 + \left(\frac{w_A}{A}\right)^2 + \frac{1}{4}\left(\frac{w_{p_1}}{p_1}\right)^2 + \frac{1}{4}\left(\frac{w_{\Delta p}}{\Delta p}\right)^2 + \frac{1}{4}\left(\frac{w_{T_1}}{T_1}\right)^2\right]^{1/2} \qquad \textbf{[d]}$$

We may now insert the numerical values for the quantities to obtain the percent uncertainty in the mass flow rate.

$$\frac{w_{\dot{m}}}{\dot{m}} = \left[\left(\frac{0.005}{0.92}\right)^2 + \left(\frac{0.001}{1.0}\right)^2 + \frac{1}{4}\left(\frac{0.5}{25}\right)^2 + \frac{1}{4}\left(\frac{0.005}{1.4}\right)^2 + \frac{1}{4}\left(\frac{2}{530}\right)^2\right]^{1/2}$$

$$= [29.5 \times 10^{-6} + 1.0 \times 10^{-6} + 1.0 \times 10^{-4} + 3.19 \times 10^{-6} + 3.57 \times 10^{-6}]^{1/2}$$

$$= [1.373 \times 10^{-4}]^{1/2} = 1.172\% \qquad \textbf{[e]}$$

Comment

The main contribution to uncertainty is the p_1 measurement with its basic uncertainty of 2 percent. Thus, to improve the overall situation the accuracy of this measurement should be attacked first. In order of influence on the flow-rate uncertainty we have:

1. Uncertainty in p_1 measurement (± 2 percent)
2. Uncertainty in value of C
3. Uncertainty in determination of T_1
4. Uncertainty in determination of Δp
5. Uncertainty in determination of A

By inspecting Eq. (*e*) we see that the first and third items make practically the whole contribution to uncertainty. The value of the uncertainty analysis in this example is that it shows the investigator how to improve the overall measurement accuracy of this technique. First, obtain a more precise measurement of p_1. Then, try to obtain a better calibration of the device, that is, a better value of C. In Chap. 7 we shall see how values of the discharge coefficient C are obtained.

UNCERTAINTY CALCULATION BY RESULT PERTURBATION. Calculate the uncertainty of the wire resistance in Example 3.1 using the result-perturbation technique.

Solution

In Example 3.1 we have already calculated the nominal resistance as 6.24 Ω. We now perturb the three variables R_0, α, and T by small amounts to evaluate the partial derivatives. We shall take

$$\Delta R_0 = 0.01 \qquad \Delta\alpha = 1 \times 10^{-5} \qquad \Delta T = 0.1$$

Then $$R(R_0 + \Delta R_0) = (6.01)[1 + (0.004)(30 - 20)] = 6.2504$$

and the derivative is approximated as

$$\frac{\partial R}{\partial R_0} \approx \frac{R(R_0 + \Delta R_0) - R}{\Delta R_0} = \frac{6.2504 - 6.24}{0.01} = 1.04$$

or the same result as in Example 3.1. Similarly,

$$R(\alpha + \Delta\alpha) = (6.0)[1 + (0.00401)(30 - 20)] = 6.2406$$

$$\frac{\partial R}{\partial \alpha} \approx \frac{R(\alpha + \Delta\alpha) - R}{\Delta\alpha} = \frac{6.2406 - 6.24}{1 \times 10^{-5}} = 60$$

$$R(T + \Delta T) = (6)[1 + (0.004)(30.1 - 20)] = 6.2424$$

$$\frac{\partial R}{\partial T} \approx \frac{R(T + \Delta T) - R}{\Delta T} = \frac{6.2424 - 6.24}{0.1} = 0.24$$

All the derivatives are the same as in Example 3.1, so the uncertainty in R would be the same, or 0.0305 Ω.

3.6 Statistical Analysis of Experimental Data

We shall not be able to give an extensive presentation of the methods of statistical analysis of experimental data; we may only indicate some of the more important methods currently employed. First, it is important to define some pertinent terms.

When a set of readings of an instrument is taken, the individual readings will vary somewhat from each other, and the experimenter may be concerned with the *mean* of all the readings. If each reading is denoted by x_i and there are n readings, the *arithmetic mean* is given by

$$x_m = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \textbf{[3.3]}$$

The *deviation* d_i for each reading is defined by

$$d_i = x_i - x_m \qquad \textbf{[3.4]}$$

We may note that the average of the deviations of all the readings is zero since

$$\bar{d}_i = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)$$

$$= x_m - \frac{1}{n}(nx_m) = 0 \qquad \textbf{[3.5]}$$

The average of the absolute values of the deviations is given by

$$|\bar{d}_i| = \frac{1}{n}\sum_{i=1}^{n} |d_i| = \frac{1}{n}\sum_{i=1}^{n} |x_i - x_m| \qquad \textbf{[3.6]}$$

Note that this quantity is not necessarily zero.

The *standard deviation* or *root-mean-square deviation* is defined by

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2} \qquad \textbf{[3.7]}$$

and the square of the standard deviation σ^2 is called the *variance*. This is sometimes called the *population* or *biased* standard deviation because it strictly applies only when a large number of samples is taken to describe the population.

In many circumstances the engineer will not be able to collect as many data points as necessary to describe the underlying population. Generally speaking, it is desired to have at least 20 measurements in order to obtain reliable estimates of standard deviation and general validity of the data. For small sets of data an *unbiased* or *sample standard deviation* is defined by

$$\sigma = \left[\frac{\sum_{i=1}^{n}(x_i - x_m)^2}{n-1}\right]^{1/2} \qquad \textbf{[3.8]}$$

Note that the factor $n - 1$ is used instead of n as in Eq. (3.7). The sample or unbiased standard deviation should be used when the underlying population is not known. However, when comparisons are made against a known population or standard, Eq. (3.7) is the proper one to use for standard deviation. An example would be the calibration of a voltmeter against a known voltage source.

There are other kinds of mean values of interest from time to time in statistical analysis. The *median* is the value that divides the data points in half. For example, if measurements made on five production resistors give 10, 12, 13, 14, and 15 kΩ, the median value would be 13 kΩ. The *arithmetic* mean, however, would be

$$R_m = \frac{10 + 12 + 13 + 14 + 15}{5} = 12.8 \text{ k}\Omega$$

In some instances it may be appropriate to divide data into quartiles and deciles also. So, when we say that a student is in the upper quartile of the class, we mean that that student's grade is among the top 25 percent of all students in the class.

Sometimes it is appropriate to use a *geometric mean* when studying phenomena which grow in proportion to their size. This would apply to certain biological processes and to growth rates in financial resources. The geometric mean is defined by

$$x_g = [x_1 \cdot x_2 \cdot x_3 \cdots x_n]^{1/n} \qquad \textbf{[3.9]}$$

As an example of the use of this concept, consider the 5-year record of a mutual fund investment:

Year	Asset Value	Rate of Increase over Previous Year
1	1000	
2	890	0.89
3	990	1.1124
4	1100	1.1111
5	1250	1.1364

The average growth rate is therefore

$$\begin{aligned} \text{Average growth} &= [(0.89)(1.1124)(1.1111)(1.1364)]^{1/4} \\ &= 1.0574 \end{aligned}$$

To see that this is indeed a valid average growth rate, we can observe that

$$(1000)(1.0574)^4 = 1250$$

CALCULATION OF POPULATION VARIABLES. The following readings are taken of a certain physical length. Compute the mean reading, standard deviation, variance, and average of the absolute value of the deviation, using the "biased" basis:

Reading	x, cm
1	5.30
2	5.73
3	6.77
4	5.26
5	4.33
6	5.45
7	6.09
8	5.64
9	5.81
10	5.75

Solution

The mean value is given by

$$x_m = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{10}(56.13) = 5.613 \text{ cm}$$

The other quantities are computed with the aid of the following table:

Reading	$d_i = x_i - x_m$	$(x_i - x_m)^2 \times 10^2$
1	−0.313	9.797
2	0.117	1.369
3	1.157	133.865
4	−0.353	12.461
5	−1.283	164.609
6	−0.163	2.657
7	0.477	22.753
8	0.027	0.0729
9	0.197	3.881
10	0.137	1.877

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2} = \left[\frac{1}{10}(3.533)\right]^{1/2} = 0.5944 \text{ cm}$$

$$\sigma^2 = 0.3533 \text{ cm}^2$$

$$|\bar{d_i}| = \frac{1}{n}\sum_{i=1}^{n}|d_i| = \frac{1}{n}\sum_{i=1}^{n}|x_i - x_m|$$

$$= \tfrac{1}{10}(4.224) = 0.4224 \text{ cm}$$

3.7 Probability Distributions

Suppose we toss a horseshoe some distance x. Even though we make an effort to toss the horseshoe the same distance each time, we would not always meet with success. On the first toss the horseshoe might travel a distance x_1, on the second toss a distance of x_2, and so forth. If one is a good player of the game, there would be more tosses which have an x distance equal to that of the objective. Also, we would expect fewer and fewer tosses for those x distances which are farther and farther away from the target. For a large number of tosses the probability that it will travel a distance is obtained by dividing the number traveling this distance by the total number of tosses. Since each x distance will vary somewhat from other x distances, we might find it advantageous to calculate the probability of a toss landing in a certain increment of x between x and $x + \Delta x$. When this calculation is made, we might get something like the situation shown in Fig. 3.1. For a good player the maximum probability is expected to surround the distance x_m designating the position of the target.

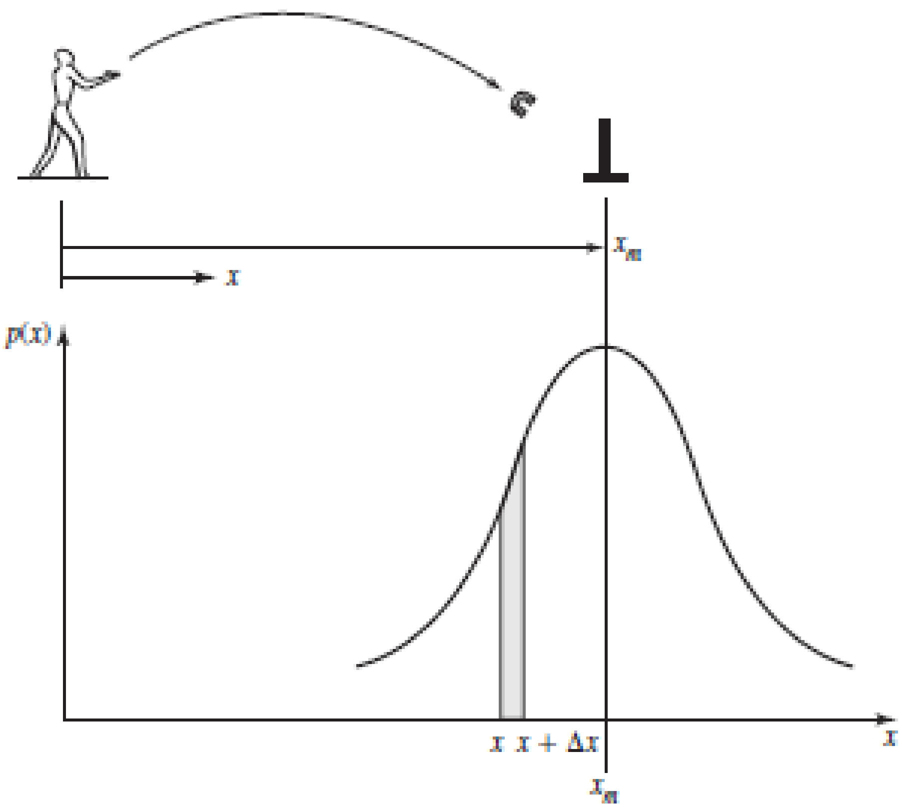

Figure 3.1 Distribution of throws for a "good" horseshoes player.

The curve shown in Fig. 3.1 is called a *probability distribution*. It shows how the probability of success in a certain event is distributed over the distance x. Each value of the ordinate $p(x)$ gives the probability that the horseshoe will land between x and $x + \Delta x$, where Δx is allowed to approach zero. We might consider the deviation from x_m as the error in the throw. If the horseshoe player has good aim, large errors are less likely than small errors. The area under the curve is unity since it is certain that the horseshoe will land somewhere.

We should note that more than one variable may be present in a probability distribution. In the case of the horseshoes player a person might throw the object an exact distance of x_m and yet to one side of the target. The sideways distance is another variable, and a large number of throws would have some distribution in this variable as well.

A particular probability distribution is the *binomial distribution*. This distribution gives the number of successes n out of N possible independent events when each event has a probability of success p. The probability that n events will succeed is given in Ref. [2] as

$$p(n) = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \qquad \textbf{[3.10]}$$

It will be noted that the quantity $(1-p)$ is the probability of failure of each independent event. Now, suppose that the number of possible independent events N is very large and the probability of occurrence of each p is very small. The calculation of the probability of n successes out of the N possible events using Eq. (3.10) would be most cumbersome because of the size of the numbers. The limit of the binomial distribution as $N \to \infty$ and $p \to 0$ such that

$$Np = a = \text{const}$$

is called the *Poisson distribution* and is given by

$$p_a(n) = \frac{a^n e^{-a}}{n!} \qquad \textbf{[3.11]}$$

The Poisson distribution is applicable to the calculation of the decay of radioactive nuclei, as we shall see in a subsequent chapter. It may be shown that the standard deviation of the Poisson distribution is

$$\sigma = \sqrt{a} \qquad \textbf{[3.12]}$$

Example 3.9 | **TOSSING A COIN—BINOMIAL DISTRIBUTION.** An unweighted coin is flipped three times. Calculate the probability of getting zero, one, two, or three heads in these tosses.

Solution

The binomial distribution applies in this case since the probability of each flip of the coin is independent of previous or successive flips. The probability of getting a head on each throw is $p = \frac{1}{2}$ and $N = 3$, while n takes on the values 0, 1, 2, and 3. The probabilities are calculated as

$$p(0) = \frac{3!}{(3!)(0!)}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$p(1) = \frac{3!}{(2!)(1!)}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$$p(2) = \frac{3!}{(1!)(2!)}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{3}{8}$$

$$p(3) = \frac{3!}{(0!)(3!)}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$$

Comment

Note that the sum of the four probabilities, that is, $\frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$ is unity or *certainty* because there are no other possibilities. Heads must come up zero, one, two, or three times in three flips. Of course, one would obtain the same result for probabilities of obtaining zero, one, two, or three tails in three flips.

Histograms

We have noted that a probability distribution like Fig. 3.1 is obtained when we observe frequency of occurrence over a large number of observations. When a limited number of observations is made and the raw data are plotted, we call the plot a *histogram*. For example, the following distribution of throws might be observed for a horseshoes player:

Distance from Target, cm	**Number of Throws**
0–10	5
10–20	15
20–30	13
30–40	11
40–50	9
50–60	8
60–70	10
70–80	6
80–90	7
90–100	5
100–110	5
110–120	3
Over 120	2
Total	99

These data are plotted in Fig. 3.2 using increments of 10 cm in Δx. The same data are plotted in Fig. 3.3 using a Δx of 20 cm. The *relative frequency*, or fraction of

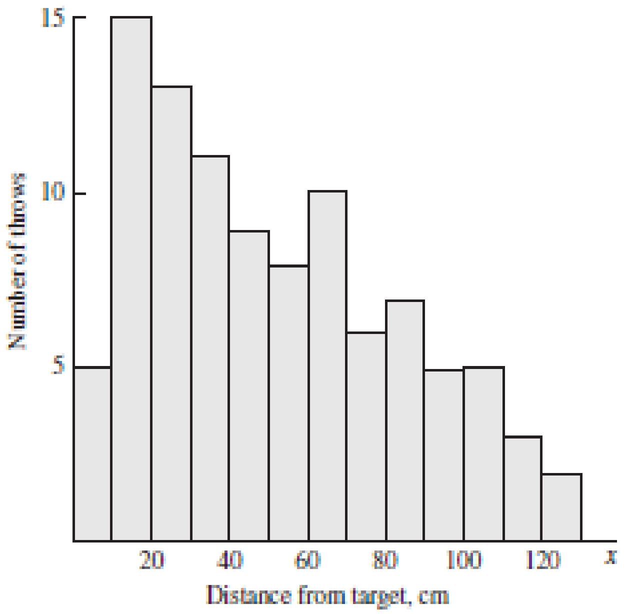

Figure 3.2 Histogram with $\Delta x = 10$ cm.

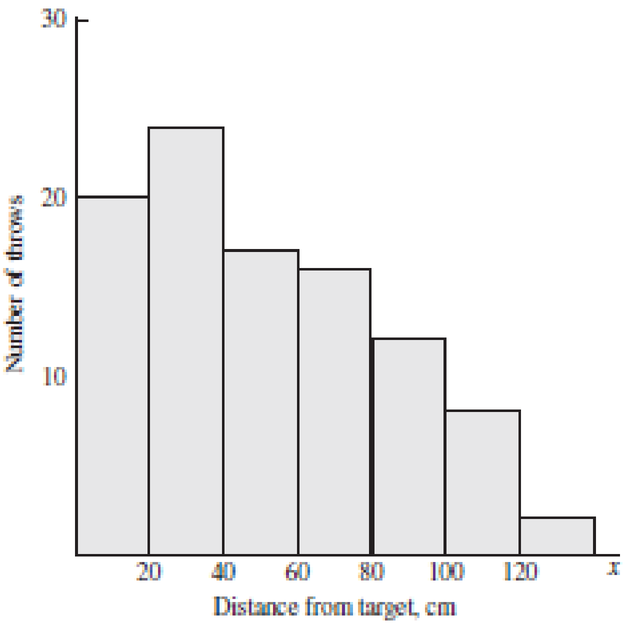

Figure 3.3 Histogram with $\Delta x = 20$ cm.

throws in each Δx increment, could also be used to convey the same information. A *cumulative frequency* diagram could be employed for these data, as shown in Fig. 3.4. If this figure had been constructed on the basis of a very large number of throws, then we could appropriately refer to the ordinate as the probability that the horseshoe will land within a distance x of the target.

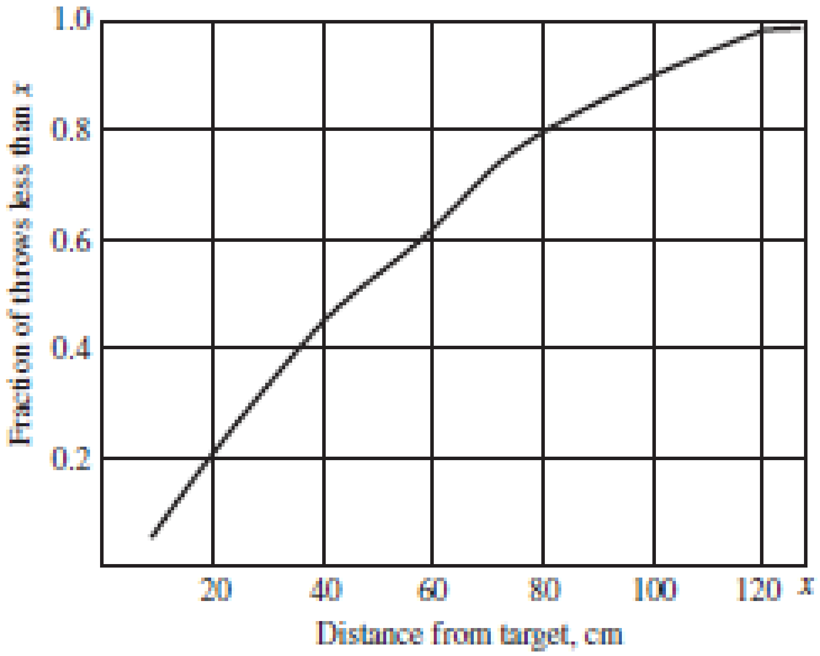

Figure 3.4 Cumulative frequency diagram.

3.8 The Gaussian or Normal Error Distribution

Suppose an experimental observation is made and some particular result is recorded. We know (or would strongly suspect) that the observation has been subjected to many random errors. These random errors may make the final reading either too large or too small, depending on many circumstances which are unknown to us. Assuming that there are many small errors that contribute to the final error and that each small error is of equal magnitude and equally likely to be positive or negative, the *gaussian* or *normal error distribution* may be derived. If the measurement is designated by x, the gaussian distribution gives the probability that the measurement will lie between x and $x + dx$ and is written

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-x_m)^2/2\sigma^2} \quad \textbf{[3.13]}$$

In this expression x_m is the mean reading and σ is the standard deviation. Some may prefer to call $P(x)$ the *probability density*. The units of $P(x)$ are those of $1/x$ since these are the units of $1/\sigma$. A plot of Eq. (3.13) is given in Fig. 3.5. Note that the most probable reading is x_m. The standard deviation is a measure of the width of the distribution curve; the larger the value of σ, the flatter the curve and hence the larger the expected error of all the measurements. Equation (3.13) is normalized so that the total area under the curve is unity. Thus,

$$\int_{-\infty}^{+\infty} P(x)\, dx = 1.0 \quad \textbf{[3.14]}$$

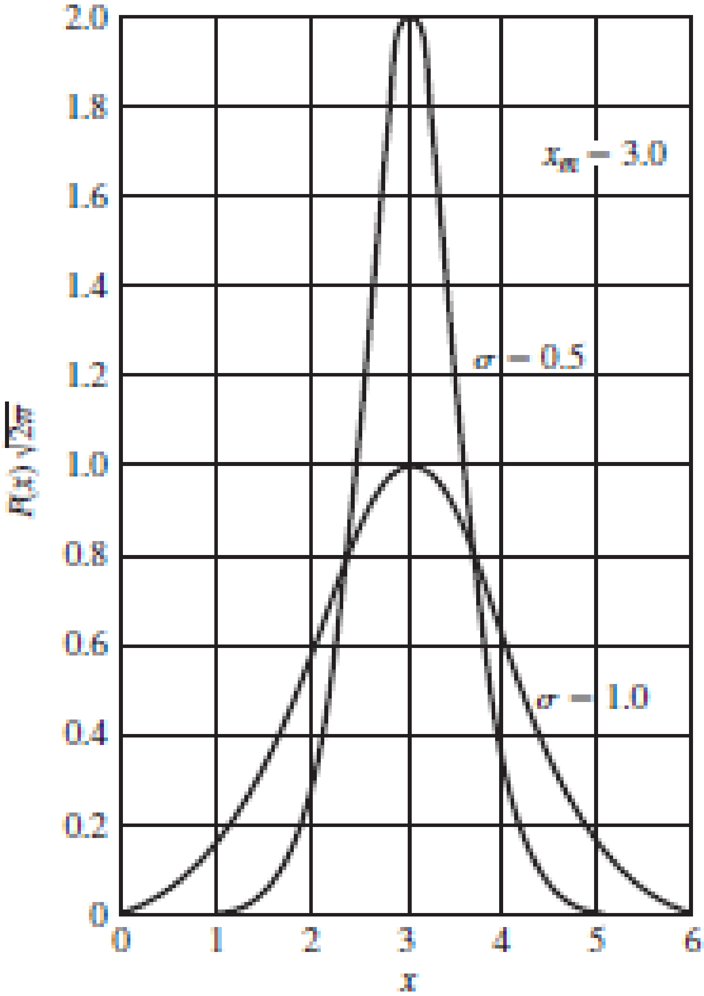

Figure 3.5 The gaussian or normal error distribution for two values of the standard deviation.

At this point we may note the similarity between the shape of the normal error curve and the expected experimental distribution for tossing horseshoes, as shown in Fig. 3.1. This is what we would expect because the good horseshoes player's throws will be bunched around the target. The better the player is at the game, the more closely the throws will be grouped around the mean and the more probable will be the mean distance x_m. Thus, in the case of the horseshoes player a smaller standard deviation would mean a larger percentage of "ringers."

We may quickly anticipate the next step in the analysis as one of trying to determine the precision of a set of experimental measurements through an application of the normal error distribution. One may ask: But how do you know that the assumptions pertaining to the derivation of the normal error distribution apply to experimental data? The answer is that for sets of data where a large number of measurements is taken, experiments indicate that the measurements do indeed follow a distribution like that shown in Fig. 3.5 when the experiment is under control. If an important parameter is not controlled, one gets just scatter, that is, no sensible distribution at all. Thus, as a matter of experimental verification, the gaussian distribution is believed to represent the *random* errors in an adequate manner for a properly controlled experiment.

By inspection of the gaussian distribution function of Eq. (3.13) we see that the maximum probability occurs at $x = x_m$, and the value of this probability is

$$P(x_m) = \frac{1}{\sigma\sqrt{2\pi}} \qquad \textbf{[3.15]}$$

It is seen from Eq. (3.15) that smaller values of the standard deviation produce larger values of the maximum probability, as would be expected in an intuitive sense. $P(x_m)$ is sometimes called a *measure of precision* of the data because it has a larger value for smaller values of the standard deviation.

We next wish to examine the gaussian distribution to determine the likelihood that certain data points will fall within a specified deviation from the mean of all the data points. The probability that a measurement will fall within a certain range x_1 of the mean reading is

$$P = \int_{x_m - x_1}^{x_m + x_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-x_m)^2/2\sigma^2}\, dx \qquad \textbf{[3.16]}$$

Making the variable substitution

$$\eta = \frac{x - x_m}{\sigma}$$

Eq. (3.16) becomes

$$P = \frac{1}{\sqrt{2\pi}} \int_{-\eta_1}^{+\eta_1} e^{-\eta^2/2}\, d\eta \qquad \textbf{[3.17]}$$

where

$$\eta_1 = \frac{x_1}{\sigma} \qquad \textbf{[3.18]}$$

Values of the gaussian normal error function

$$\frac{1}{\sqrt{2\pi}} e^{-\eta^2/2}$$

and integrals of the gaussian function corresponding to Eq. (3.17) are given in Tables 3.1 and 3.2.

If we have a sufficiently large number of data points, the error for each point should follow the gaussian distribution and we can determine the probability that certain data fall within a specified deviation from the mean value. Example 3.10 illustrates the method of computing the chances of finding data points within one or two standard deviations from the mean. Table 3.3 gives the chances for certain deviations from the mean value of the normal distribution curve.

Values of the function $(1/\sqrt{2\pi})e^{-\eta^2/2}$ for different values of the argument η. Each figure in the body of the table is preceded by a decimal point.

η	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	39894	39892	39886	39876	39862	39844	39822	39797	39767	39733
0.1	39695	39654	39608	39559	39505	39448	39387	39322	39253	39181
0.2	39104	39024	38940	38853	38762	38667	38568	38466	38361	38251
0.3	38139	38023	37903	37780	37654	37524	37391	37255	37115	36973
0.4	36827	36678	36526	36371	36213	36053	35889	35723	35553	35381
0.5	35207	35029	34849	34667	34482	34294	34105	33912	33718	33521
0.6	33322	33121	32918	32713	32506	32297	32086	31875	31659	31443
0.7	31225	31006	30785	30563	30339	30114	29887	29658	29430	29200
0.8	28969	28737	28504	28269	28034	27798	27562	27324	27086	26848
0.9	26609	26369	36129	25888	25647	25406	25164	24923	24681	24439
1.0	24197	23955	23713	23471	23230	22988	22747	22506	22265	22025
1.1	21785	21546	21307	21069	20831	20594	20357	20121	19886	19652
1.2	19419	19186	18954	18724	18494	18265	18037	17810	17585	17360
1.3	17137	16915	16694	16474	16256	16038	15822	15608	15395	15183
1.4	14973	14764	14556	14350	14146	13943	13742	13542	13344	13147
1.5	12952	12758	12566	12376	12188	12001	11816	11632	11450	11270
1.6	11092	10915	10741	10567	10396	10226	10059	09893	09728	09566
1.7	09405	09246	09089	08933	08780	08628	08478	08329	08183	08038
1.8	07895	07754	07614	07477	07341	07206	07074	06943	06814	06687
1.9	06562	06438	06316	06195	06077	05959	05844	05730	05618	05508
2.0	05399	05292	05186	05082	04980	04879	04780	04682	04586	04491
2.1	04398	04307	04217	04128	04041	03955	03871	03788	03706	03626
2.2	03547	03470	03394	03319	03246	03174	03103	03034	02965	02898
2.3	02833	02768	02705	02643	02582	02522	02463	02406	02349	02294
2.4	02239	02186	02134	02083	02033	01984	01936	01888	01842	01797
2.5	01753	01709	01667	01625	01585	01545	01506	01468	01431	01394
2.6	01358	01323	01289	01256	01223	01191	01160	01130	01100	01071
2.7	01042	01014	00987	00961	00935	00909	00885	00861	00837	00814
2.8	00792	00770	00748	00727	00707	00687	00668	00649	00631	00613
2.9	00595	00578	00562	00545	00530	00514	00499	00485	00470	00457
3.0	00443									
3.5	008727									
4.0	0001338									
4.5	0000160									
5.0	000001487									

Table 3.2 Integrals of the gaussian normal error function

Values of the integral $(1/\sqrt{2\pi})\int_0^{\eta_1} e^{-\eta^2/2}d\eta$ are given for different values of the argument η_1. It may be observed that

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta_1}^{+\eta_1} e^{-\eta^2/2}d\eta = 2\frac{1}{\sqrt{2\pi}}\int_0^{\eta_1} e^{-\eta^2/2}d\eta$$

The values are related to the error function since

$$\text{erf } \eta_1 = \frac{1}{\sqrt{\pi}}\int_{-\eta_1}^{+\eta_1} e^{-\eta^2}d\eta$$

so that the tabular values are equal to $\frac{1}{2}$ erf $(\eta_1/\sqrt{2})$. Each figure in the body of the table is preceded by a decimal point.

η_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0000	00399	00798	01197	01595	01994	02392	02790	03188	03586
0.1	03983	04380	04776	05172	05567	05962	06356	06749	07142	07355
0.2	07926	08317	08706	09095	09483	09871	10257	10642	11026	11409
0.3	11791	12172	12552	12930	13307	13683	14058	14431	14803	15173
0.4	15554	15910	16276	16640	17003	17364	17724	18082	18439	18793
0.5	19146	19497	19847	20194	20450	20884	21226	21566	21904	22240
0.6	22575	22907	23237	23565	23891	24215	24537	24857	25175	25490
0.7	25084	26115	26424	26730	27035	27337	27637	27935	28230	28524
0.8	28814	29103	29389	29673	29955	30234	30511	30785	31057	31327
0.9	31594	31859	32121	32381	32639	32894	33147	33398	33646	33891
1.0	34134	34375	34614	34850	35083	35313	35543	35769	35993	36214
1.1	36433	36650	36864	37076	37286	37493	37698	37900	38100	38298
1.2	38493	38686	38877	39065	39251	39435	39617	39796	39973	40147
1.3	40320	40490	40658	40824	40988	41198	41308	41466	41621	41774
1.4	41924	42073	42220	42364	42507	42647	42786	42922	43056	43189
1.5	43319	43448	43574	43699	43822	43943	44062	44179	44295	44408
1.6	44520	44630	44738	44845	44950	45053	45154	45254	45352	45449
1.7	45543	45637	45728	45818	45907	45994	46080	46164	46246	46327
1.8	46407	46485	46562	46638	46712	46784	46856	46926	46995	47062
1.9	47128	47193	47257	47320	47381	47441	47500	47558	47615	47670
2.0	47725	47778	47831	47882	47932	47962	48030	48077	48124	48169
2.1	48214	48257	48300	48341	48382	48422	48461	48500	48537	48574
2.2	48610	48645	48679	48713	48745	48778	48809	48840	48870	48899
2.3	48928	48956	48983	49010	49036	49061	49086	49111	49134	49158
2.4	49180	49202	49224	49245	49266	49286	49305	49324	49343	49361
2.5	49379	49296	49413	49430	49446	49461	49477	49492	49506	49520
2.6	49534	49547	49560	49573	49585	49598	49609	49621	49632	49643
2.7	49653	49664	49674	49683	49693	49702	49711	49720	49728	49736
2.8	49744	49752	49760	49767	49774	49781	49788	49795	49801	49807
2.9	49813	49819	49825	49831	49836	49841	49846	49851	49856	49861
3.0	49865									
3.5	4997674									
4.0	4999683									
4.5	4999966									
5.0	4999997133									

Table 3.3 Chances for deviations from mean value of normal distribution curve

Deviation	Chances of Results Falling within Specified Deviation
$\pm 0.6745\sigma$	1–1
σ	2.15–1
2σ	21–1
3σ	369–1

Example 3.10 **PROBABILITY FOR DEVIATION FROM MEAN VALUE.** Calculate the probabilities that a measurement will fall within one, two, and three standard deviations of the mean value and compare them with the values in Table 3.3.

Solution

We perform the calculation using Eq. (3.17) with $\eta_1 = 1$, 2, and 3. The values of the integral may be obtained from Table 3.2. We observe that

$$\int_{-\eta_1}^{+\eta_1} e^{-\eta^2/2}\, d\eta = 2\int_0^{\eta_1} e^{-\eta^2/2}\, d\eta$$

so that

$$P(1) = (2)(0.34134) = 0.6827$$

$$P(2) = (2)(0.47725) = 0.9545$$

$$P(3) = (2)(0.49865) = 0.9973$$

Using the odds given in Table 3.3, we would calculate the probabilities as

$$P(1) = \frac{2.15}{2.15+1} = 0.6827$$

$$P(2) = \frac{21}{21+1} = 0.9545$$

$$P(3) = \frac{369}{369+1} = 0.9973$$

Comment

This example shows how the concept of probability in the gaussian distribution is related to the "odds" concept mentioned in the previous discussion of uncertainty specifications.

Confidence Interval and Level of Significance

The *confidence interval* expresses the probability that the mean value will lie within a certain number of σ values and is given by the symbol z. Thus,

$$\bar{x} = \bar{x} \pm z\sigma \qquad (\%\ \text{confidence level})$$

Table 3.4

Confidence Interval	Confidence Level, %	Level of Significance, %
3.30	99.9	0.1
3.0	99.7	0.3
2.57	99.0	1.0
2.0	95.4	4.6
1.96	95.0	5.0
1.65	90.0	10.0
1.0	68.3	31.7

and using the procedure of Example 3.10, the confidence level in percent could be expressed as in Table 3.4. For small data samples z should be replaced by

$$\Delta = \frac{z\sigma}{\sqrt{n}} \qquad \textbf{[3.19]}$$

We thus expect that the mean value will lie within $\pm 2.57\sigma$ with less than 1 percent error (confidence level of 99 percent). The *level of significance* is 1 minus the confidence level. Thus, for $z = 2.57$ the level of significance is 1 percent.

Example 3.11

DETERMINATION OF NUMBER OF MEASUREMENTS TO ASSURE A SIGNIFICANCE LEVEL. A certain steel bar is measured with a device which has a known precision of ± 0.5 mm when a large number of measurements is taken. How many measurements are necessary to establish the mean length $\bar{x}$ with a 5 percent level of significance such that

$$\bar{x} = \bar{x} \pm 0.2 \text{ mm}$$

Solution

For a large number of measurements the 5 percent level of significance is obtained at $z = 1.96$ and for the population here

$$\Delta = \frac{z\sigma}{\sqrt{n}} = 0.2 \text{ mm} = \frac{(1.96)(0.5 \text{ mm})}{\sqrt{n}}$$

which yields

$$n = 24.01$$

So, for 25 measurements or more we could state with a confidence level of 95 percent that the population mean value will be within ± 0.2 mm of the sample mean value.

Example 3.12

POWER SUPPLY. A certain power supply is stated to provide a constant voltage output of 10.0 V within ± 0.1 V. The output is assumed to have a normal distribution. Calculate the probability that a single measurement of voltage will lie between 10.1 and 10.2 V.

Solution

For this problem $\sigma = \pm 0.1$ V. The probability that the voltage will lie between 10.0 and 10.1 V ($+1\sigma$) is, from Table 3.2,

$$P(+0.1) = 0.34134$$

while the probability it will lie between 10.0 and 10.2 V ($+2\sigma$) is

$$P(+0.2) = 0.47725$$

The probability that it will lie between 10.1 and 10.2 V is therefore

$$P(10.1 \text{ to } 10.2) = 0.47725 - 0.34134 = 0.13591$$

Chauvenet's Criterion

It is a rare circumstance indeed when an experimenter does not find that some of the data points look bad and out of place in comparison with the bulk of the data. The experimenter is therefore faced with the task of deciding if these points are the result of some gross experimental blunder and hence may be neglected or if they represent some new type of physical phenomenon that is peculiar to a certain operating condition. The engineer cannot just throw out those points that do not fit with expectations—there must be some consistent basis for elimination.

Suppose n measurements of a quantity are taken and n is large enough that we may expect the results to follow the gaussian error distribution. This distribution may be used to compute the probability that a given reading will deviate a certain amount from the mean. We would not expect a probability much smaller than $1/n$ because this would be unlikely to occur in the set of n measurements. Thus, if the probability for the observed deviation of a certain point is less than $1/n$, a suspicious eye would be cast at that point with an idea toward eliminating it from the data. Actually, a more restrictive test is usually applied to eliminate data points. It is known as *Chauvenet's criterion*[1] and specifies that a reading may be rejected if the probability of obtaining the particular deviation from the mean is less than $1/2n$. Table 3.5 lists values of the ratio of deviation to standard deviation for various values of n according to this criterion with Fig. 3.6 furnishing a graphical representation.

In applying Chauvenet's criterion to eliminate dubious data points, one first calculates the mean value and standard deviation using all data points. The deviations of the individual points are then compared with the standard deviation in accordance with the information in Table 3.5 (or by a direct application of the criterion), and the dubious points are eliminated. For the final data presentation a new mean value and standard deviation are computed with the dubious points eliminated from the

Table 3.5 Chauvenet's criterion for rejecting a reading

Number of Readings, n	Ratio of Maximum Acceptable Deviation to Standard Deviation, d_{max}/σ
3	1.38
4	1.54
5	1.65
6	1.73
7	1.80
10	1.96
15	2.13
25	2.33
50	2.57
100	2.81
300	3.14
500	3.29
1000	3.48

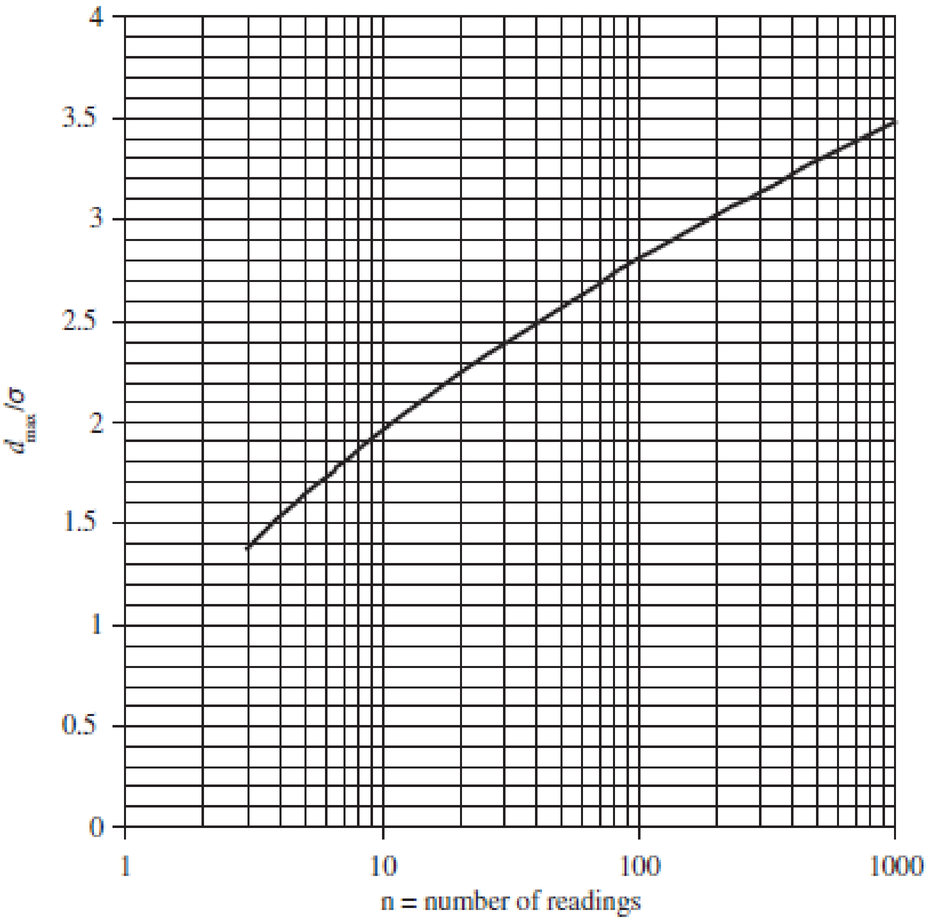

Figure 3.6 Chauvenet's criterion.

calculation. Note that Chauvenet's criterion might be applied a second or third time to eliminate additional points; but this practice is unacceptable, and only the first application may be used.

Example 3.13 **APPLICATION OF CHAUVENET'S CRITERION.** Using Chauvenet's criterion, test the data points of Example 3.7 for possible inconsistency. Eliminate the questionable points and calculate a new standard deviation for the adjusted data.

Solution

The best estimate of the standard deviation is given in Example 3.8 as 0.627 cm. We first calculate the ratio d_i/σ and eliminate data points in accordance with Table 3.5.

Reading	d_i/σ
1	0.499
2	0.187
3	1.845
4	0.563
5	2.046
6	0.260
7	0.761
8	0.043
9	0.314
10	0.219

In accordance with Table 3.5, we may eliminate only point number 5. When this point is eliminated, the new mean value is

$$x_m = \tfrac{1}{9}(51.80) = 5.756 \text{ cm}$$

The new value of the standard deviation is now calculated with the following table:

Reading	$d_i = x_i - x_m$	$(x_i - x_m)^2 \times 10^2$
1	−0.456	20.7936
2	−0.026	0.0676
3	1.014	102.8196
4	−0.496	24.602
6	−0.306	9.364
7	0.334	11.156
8	−0.116	1.346
9	0.054	0.292
10	−0.006	0.0036

$$\sigma = \left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2} = \left[\tfrac{1}{8}(1.7044)\right]^{1/2} = (0.213)^{1/2} = 0.4615 \text{ cm}$$

Thus, by the elimination of the one point the standard deviation has been reduced from 0.627 to 0.462 cm. This is a 26.5 percent reduction.

Comment

Please note that for the revised calculation of standard deviation a new mean value must be computed leaving out the excluded data point.

The Chauvenet's criterion we have applied in this example is

$$d_{\max}/\sigma = 1.96 \text{ for } n = 10$$

This value may be calculated directly from the gaussian distribution shown in Table 3.2 in the following way. The criterion is that the probability of a point lying outside the normal distribution should not exceed $1/2n$ or $1/20$. The probability of the point lying *inside* the normal distribution would then be

$$P(n) = 1 - 1/20 = 0.95$$

The entry point for Table 3.2 is half this value or 0.475. We obtain

$$\eta = 1.96$$

which agrees, of course, with Table 3.5 and Fig. 3.6.