Chapter 4
Nominal and Effective Interest Rates

4.1. Nominal and Effective Interest Rate Statements

A nominal interest rate $r$ is an interest rate that does not account for compounding.

$$r = \text{interest rate per time period} \times \text{number of periods}$$

A nominal rate may be calculated for any time period longer than the time period stated. For example, the interest rate of 1.5% per month is the same as each of the following nominal rates.

<table>
<thead>
<tr>
<th>Time Period</th>
<th>Nominal Rate by Equation [4.1]</th>
<th>What This Is</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 months</td>
<td>$1.5 \times 24 = 36%$</td>
<td>Nominal rate per 2 years</td>
</tr>
<tr>
<td>12 months</td>
<td>$1.5 \times 12 = 18%$</td>
<td>Nominal rate per 1 year</td>
</tr>
<tr>
<td>6 months</td>
<td>$1.5 \times 6 = 9%$</td>
<td>Nominal rate per 6 months</td>
</tr>
<tr>
<td>3 months</td>
<td>$1.5 \times 3 = 4.5%$</td>
<td>Nominal rate per 3 months</td>
</tr>
</tbody>
</table>

Note that none of these rates mention anything about compounding of interest; they are all of the form “$r \% \text{ per time period}$”.

After the nominal rate has been calculated, the compounding period (CP) must be included in the interest rate statement. As an illustration, again consider the nominal rate of 1.5% per month. If we define the CP as 1 month, the nominal rate statement is 18% per year, compounded monthly, or 4.5% per quarter, compounded monthly.

An effective interest rate $i$ is a rate wherein the compounding of interest is taken into account.

For example, 10% per year, compounded monthly, or 12% per year, compounded weekly. If the CP is not mentioned, it is the same as the time period mentioned with the interest rate. For example, an interest rate of “1.5% per month” means that interest is
compounded monthly. Therefore, the equivalent effective rate statement is 1.5% per month, compounded monthly.

All of the following are effective interest rate statements because either they state they are effective or the compounding period is not mentioned. In the latter case, the CP is the same as the time period of the interest rate.

<table>
<thead>
<tr>
<th>Statement</th>
<th>CP</th>
<th>What This Is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 10% \text{ per year} )</td>
<td>CP not stated; CP = year</td>
<td>Effective rate per year</td>
</tr>
<tr>
<td>( i = \text{effective } 10% \text{ per year, compounded monthly} )</td>
<td>CP stated; CP = month</td>
<td>Effective rate per year</td>
</tr>
<tr>
<td>( i = 1\frac{1}{2}% \text{ per month} )</td>
<td>CP not stated; CP = month</td>
<td>Effective rate per month</td>
</tr>
<tr>
<td>( i = \text{effective } 1\frac{1}{2}% \text{ per month, compounded monthly} )</td>
<td>CP stated; CP = month</td>
<td>Effective rate per month and compounded monthly are redundant</td>
</tr>
<tr>
<td>( i = \text{effective } 3% \text{ per quarter, compounded daily} )</td>
<td>CP stated; CP = day</td>
<td>Effective rate per quarter</td>
</tr>
</tbody>
</table>

All interest formulas, factors and tabulated values must use an effective interest rate to properly account for the time value of money.

**Interest period (t):** The period of time over which the interest is expressed. This is the \( t \) in the statement of \( r\% \text{ per time period } t \), for example, 1% per month.

**Compounding period (CP):** The shortest time unit over which interest is charged or earned. For example, 8% per year, compounded monthly.

**Compounding frequency (m):** The number of times that compounding occurs within the interest period \( t \).

\[
\text{Effective rate per CP} = \frac{r\% \text{ per time period } t}{m \text{ compounding periods per } t} = \frac{r}{m}
\]

**Example 4.1**

Three different bank loan rates for electric generation equipment are listed below. Determine the effective rate on the basis of the compounding period for each rate.

(a) 9% per year, compounded quarterly.
(b) 9% per year, compounded monthly.

(c) 4.5% per 6 months, compounded weekly?

**Solution**

<table>
<thead>
<tr>
<th>Nominal r% per t</th>
<th>Compounding Period (CP)</th>
<th>Compounding Frequency (m)</th>
<th>Effective Rate per CP ( \left( \frac{r}{m} \right) )</th>
<th>Distribution over Time Period t</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 9% per year</td>
<td>Quarter</td>
<td>4</td>
<td>2.25%</td>
<td>Quarter</td>
</tr>
<tr>
<td>(b) 9% per year</td>
<td>Month</td>
<td>12</td>
<td>0.75%</td>
<td>Month</td>
</tr>
<tr>
<td>(c) 4.5% per 6 months</td>
<td>Week</td>
<td>26</td>
<td>0.173%</td>
<td>Week</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TABLE 4-1</th>
<th>Various Ways to Express Nominal and Effective Interest Rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Format of Rate Statement</td>
<td>Example of Statement</td>
</tr>
<tr>
<td>Nominal rate stated, compounding period stated</td>
<td>8% per year, compounded quarterly</td>
</tr>
<tr>
<td>Effective rate stated</td>
<td>Effective 8.243% per year, compounded quarterly</td>
</tr>
<tr>
<td>Interest rate stated, no compounding period stated</td>
<td>8% per year</td>
</tr>
</tbody>
</table>

**4.2. Effective Annual Interest Rates**

The symbols used for nominal and effective interest rates are:

- \( r \) = nominal interest rate per year
- \( CP \) = time period for each compounding
- \( m \) = number of compounding periods per year
- \( i \) = effective interest rate per compounding period = \( \frac{r}{m} \)
- \( i_a \) = effective interest rate per year
The effective annual interest rate formula is:

\[ i_a = (1 + i)^m - 1 \]

This equation calculates the effective annual interest rate \( i_a \) for any number of compounding periods per year when \( i \) is the rate for one compounding period.

If the effective annual rate \( i_a \) and compounding frequency \( m \) are known, the previous equation can be solved for \( i \) to determine the effective interest rate per compounding period.

\[ i = (1 + i_a)^{1/m} - 1 \]
Example 4.2:

Janice is an engineer with Southwest Airlines. She purchased Southwest stock for $6.90 per share and sold it exactly 1 year later for $13.14 per share. What did she earn in terms of (a) effective annual rate and (b) effective rate for quarterly compounding, and for monthly compounding?
Solution

(a) The effective annual rate of return \( i_a \) has a compounding period of 1 year, since the stock purchase and sales dates are exactly 1 year apart. Based on the purchase price of $6.90 per share

\[
i_a = \frac{\text{amount of increase per 1 year}}{\text{original price}} \times 100\% = \frac{6.24}{6.90} \times 100\% = 90.43\% \text{ per year}
\]

(b)

Quarter: \( m = 4 \) times per year \( i = (1.9043)^{\frac{1}{4}} - 1 = 1.17472 - 1 = 0.17472 \)

This is 17.472% per quarter, compounded quarterly.

Month: \( m = 12 \) times per year \( i = (1.9043)^{\frac{1}{12}} - 1 = 1.05514 - 1 = 0.05514 \)

This is 5.514% per month, compounded monthly.

4.3. Effective Interest Rates for Any Time Period

\[
\text{Effective } i \text{ per time period } = (1 + \frac{r}{m})^m - 1
\]

where \( i \) = effective rate for specified time period (say, semiannual)

\( r \) = nominal interest rate for same time period (semiannual)

\( m \) = number of times interest is compounded per stated time period (times per 6 months)

Example 4.3:

There are three bids as follows:

Bid 1: 9% per year, compounded quarterly

Bid 2: 3% per quarter, compounded quarterly

Bid 3: 8.8% per year, compounded monthly

(a) Determine the effective rate for each bid on the basis of semiannual periods. (b) What are the effective annual rates? These are to be a part of the final bid selection. (c) Which bid has the lowest effective annual rate?
Solution (a)

\[ r = 9\% \text{ per year} = 4.5\% \text{ per 6 months} \]
\[ m = 2 \text{ quarters per 6 months} \]
\[ \text{Effective } i\% \text{ per 6 months} = \left(1 + \frac{0.045}{2}\right)^2 - 1 = 1.0455 - 1 = 4.55\% \]

(b)

\[ r = 9\% \text{ per year} \quad m = 4 \text{ quarters per year} \]
\[ \text{Effective } i\% \text{ per year} = \left(1 + \frac{0.09}{4}\right)^4 - 1 = 1.0931 - 1 = 9.31\% \]

(c) Bid 3 includes the lowest effective annual rate of 9.16\%, which is equivalent to an effective semiannual rate of 4.48\% when interest is compounded monthly.

Example 4.4:

A dot-com company plans to place money in a new venture capital fund that currently returns 18\% per year, compounded daily. What effective rate is this yearly and semiannually?

Solution

\[ \text{Effective } i\% \text{ per year} = \left(1 + \frac{0.18}{365}\right)^{365} - 1 = 19.716\% \]

\[ \text{Effective } i\% \text{ per 6 months} = \left(1 + \frac{0.09}{182}\right)^{182} - 1 = 9.415\% \]
4.4. Equivalence Relations: Payment Period and Compounding Period

The payment period (PP) is the length of time between cash flows (inflows or outflows). It is common that the lengths of the payment period and the compounding period (CP) do not coincide. It is important to determine if PP = CP, PP > CP, or PP < CP.

![Diagram showing payment period and compounding period]

4.5. Equivalence Relations: Single Amounts with PP ≥ CP

If the rate is 8% per year, PP = CP = 1 year.

If the rate is 10% per year, compounded quarterly, then PP is 1 year, CP is 1 quarter or 3 months, and PP > CP.

There are two methods to determine $i$ and $n$ for P/F and F/P factors:
Method 1: Determine the effective interest rate over the compounding period CP, and set $n$ equal to the number of compounding periods between P and F. The relations to calculate P and F are:

\[ P = F \left( \frac{P}{F}, \text{effective } i\% \text{ per CP, total number of periods } n \right) \]

\[ F = P \left( \frac{F}{P}, \text{effective } i\% \text{ per CP, total number of periods } n \right) \]

**Example 4.5:**

Assume that the interest rate is nominal 15% per year, compounded monthly. Here CP is 1 month. To find P or F over a 2-year span, calculate the effective monthly rate of 15%/12 = 1.25% and the total months of 2 * 12 = 24. Then \( (P/F,1.25\%,24) = 0.7422 \).

Method 2: Determine the effective interest rate for the time period $t$ of the nominal rate, and set $n$ equal to the total number of periods, using this same time period.

\[ \text{Effective } i\% \text{ per year} = \left( 1 + \frac{0.15}{12} \right)^{12} - 1 = 16.076\% \]

$n = 2$ years, and \( (P/F,16.076\%,2) = 0.7422 \). The P/F factor is the same by both methods.

**Example 4.6:**

Over the past 10 years, Gentrack has placed varying sums of money into a special capital accumulation fund. The company sells compost produced by garbage-to-compost plants in the United States and Vietnam. The cash flow diagram in $1000 units is the following. Find the amount in the account now (after 10 years) at an interest rate of 12% per year, compounded semiannually.
Solution:

Method 1: Semiannual rate of 6% per 6-month period. There are \( n = 2 \times \) number of years semiannual periods for each cash flow.

\[
F = 1000\left(\frac{F}{P,6\%,20}\right) + 3000\left(\frac{F}{P,6\%,12}\right) + 1500\left(\frac{F}{P,6\%,8}\right) \\
= 1000(3.2071) + 3000(2.0122) + 1500(1.5938) \\
= 11,634 \text{ ($11.634 million$)}
\]

Method 2:

\[
\text{Effective } i\% \text{ per year } = \left(1 + \frac{0.12}{2}\right)^2 - 1 = 12.36\%
\]

\[
F = 1000\left(\frac{F}{P,12.36\%,10}\right) + 3000\left(\frac{F}{P,12.36\%,6}\right) + 1500\left(\frac{F}{P,12.36\%,4}\right) \\
= 1000(3.2071) + 3000(2.0122) + 1500(1.5938) \\
= 11,634 \text{ ($11.634 million$)}
\]
4.6. **Equivalence Relations: Series with PP ≤ CP**

When cash flows involve a series (i.e., A, G, g) and the payment period equals or exceeds the compounding period in length:

- Find the effective $i$ per payment period.
- Determine $n$ as the total number of payment periods.

**Example 4.6:**

For the past 7 years, Excelon Energy has paid $500 every 6 months for a software maintenance contract. What is the equivalent total amount after the last payment, if these funds are taken from a pool that has been returning 8% per year, compounded quarterly?

**Solution:**
F = A(F/A,4.04%,14) = 500(18.3422) = $9171.09.

Example 4.7:

The Scott and White Health Plan (SWHP) has purchased a robotized prescription fulfillment system for faster and more accurate delivery to patients with stable, pill-form medication for chronic health problems, such as diabetes, thyroid, and high blood pressure. Assume this high volume system costs $3 million to install and an estimated $200,000 per year for all materials, operating, personnel, and maintenance costs. The expected life is 10 years. An SWHP biomedical engineer wants to estimate the total revenue requirement for each 6-month period that is necessary to recover the investment, interest, and annual costs. Find this semiannual $A$ value, if capital funds are evaluated at 8% per year, using two different compounding periods:

Rate 1. 8% per year, compounded semiannually.

Rate 2. 8% per year, compounded monthly.

Solution:
Rate 1. 8% per year, compounded semiannually.

PP = CP at 6 months; find the effective rate per semiannual period. Effective semiannual 
\[ i = \frac{8\%}{2} = 4\% \text{ per 6 months, compounded semiannually.} \]

Number of semiannual periods \( n = 2(10) = 20 \).

Calculate \( P \), using the \( P/F \) factor for \( n = 2, 4, \ldots, 20 \) periods because the costs are annual, 
not semiannual. Then use the \( A/P \) factor over 20 periods to find the semiannual \( A \).

\[
P = 3,000,000 + 200,000 \left[ \sum_{k=3,4}^{20} (P/F,4\%,k) \right] \\
= 3,000,000 + 200,000(6.6620) = 4,332,400 \\
A = \frac{4,332,400}{A/P,4\%,20} = \$318,778
\]

Rate 2. 8% per year, compounded monthly.

The PP is 6 months, but the CP is now monthly; therefore, PP = CP. To find the effective 
semiannual rate, \( r = 4\% \) and \( m = 6 \) months per semiannual period.
4.7. **Equivalence Relations: Single Amounts and Series with PP < CP**

If a person deposits money each month into a savings account where interest is compounded quarterly, do all the monthly deposits earn interest before the next quarterly compounding time? If a person's credit card payment is due with interest on the 15th of the month, and if the full payment is made on the 1st, does the financial institution reduce the interest owed, based on early payment? The usual answers are no. These are examples of PP < CP.

For a no-interperiod-interest policy, negative cash flows (deposits or payments) are all regarded as made at the end of the compounding period, and positive cash flows (receipts or withdrawals) are all regarded as made at the beginning.

**Example 4.8:**

Consider the following cash flow diagram in $1000. How much is future value at the end of the year, if the interest rate is 12% per year, compounded quarterly?
Solution:

Calculate the F value at $12%/4 = 3\%$ per quarter.

\[
F = 1000[-150(F/P,3\%,4) - 200(F/P,3\%,3) + (-175 + 180)(F/P,3\%,2) + 165(F/P,3\%,1) - 50]
\]

\[
= -$262,111
\]

If PP < CP and interperiod compounding is earned, then the cash flows are not moved, and the equivalent P, F, or A values are determined using the effective interest rate per payment period.

For example, weekly cash flows and quarterly compounding require that $m = 1/13$ of a quarter. When the nominal rate is 12\% per year, compounded quarterly (the same as 3\% per quarter, compounded quarterly), the effective rate per PP is:
Effective weekly \( i \% = (1.03)^{1/13} - 1 = 0.228\% \) per week

### 4.8. Effective Interest Rate for Continuous Compounding

Continuous compounding is present when the duration of the compounding period becomes infinitely small and \( m \), the number of times interest is compounded per period, becomes infinite. Businesses with large numbers of cash flows each day consider the interest to be continuously compounded for all transactions.

\[
\lim_{m \to \infty} i = \lim_{m \to \infty} \left(1 + \left(\frac{r}{m}\right)\right)^m - 1 = \lim_{h \to \infty} \left(1 + \frac{1}{h}\right)^{rh} - 1 = \lim_{h \to \infty} \left[\left(1 + \frac{1}{h}\right)^h\right]^r - 1
\]

\[
i = e^r - 1
\]

**Example 4.9:**

(a) For an interest rate of 18\% per year, compounded continuously, calculate the effective monthly and annual interest rates.

(b) An investor requires an effective return of at least 15\%. What is the minimum annual nominal rate that is acceptable for continuous compounding?

**Solution:**

(a) The nominal monthly rate is \( r = 18\%/12 = 1.5\% \), or 0.015 per month. The effective monthly rate is

\[
i \% \text{ per month} = e^r - 1 = e^{0.015} - 1 = 1.511\%
\]

Similarly, the effective annual rate using \( r = 0.18 \) per year is

\[
i \% \text{ per year} = e^r - 1 = e^{0.18} - 1 = 19.722\%
\]
(b) \( e^r - 1 = 0.15 \). \( e^r = 1.15 \) \( \ln e^r = \ln 1.15 \) \( r = 0.13976 \)

**Example 4.10:**

Engineers Marci and Suzanne both invest $5000 for 10 years at 10% per year. Compute the future worth for both individuals if Marci receives annual compounding and Suzanne receives continuous compounding.

**Solution:**

**Marci:** For annual compounding, the future worth is:

\[
F = P(F/P,10\%,10) = 5000(2.5937) = $12,969
\]

**Suzanne:** first find the effective \( i \) per year for use in the F/P factor.

Effective \( i \) % = \( e^{0.10} - 1 \) = 10.517%

\[
F = P(F/P,10.517\%,10) = 5000(2.7183) = $13,591
\]

**4.9. Interest Rates That Vary over Time**

**Example 4.11:**

CE, Inc., leases large earth tunneling equipment. The net profit from the equipment for each of the last 4 years has been decreasing, as shown below. Also shown are the annual rates of return on invested capital. The return has been increasing. Determine the present worth \( P \) and equivalent uniform series \( A \) of the net profit series. Take the annual variation of rates of return into account.

<table>
<thead>
<tr>
<th>Year</th>
<th>Net Profit</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$70,000</td>
<td>$70,000</td>
<td>$35,000</td>
<td>$25,000</td>
<td></td>
</tr>
<tr>
<td>Annual Rate</td>
<td>7%</td>
<td>7%</td>
<td>9%</td>
<td>10%</td>
<td></td>
</tr>
</tbody>
</table>
Solution:

\[
P = \left[70(P/A, 7\%, 2) + 35(P/F, 7\%, 2)(P/F, 9\%, 1)
+ 25(P/F, 7\%, 2)(P/F, 9\%, 1)(P/F, 10\%, 1)\right](1000)
= \left[70(1.8080) + 35(0.8013) + 25(0.7284)\right](1000)
= $172,816
\]

$172,816 = A[(1.8080 + (0.8013 + (0.7284))] = A[3.3377]
\]

\[A = $51,777 \text{ per year}\]