CMPE471 – Tutorial 1

Q1. Using the principle of mathematical induction, prove that:

a: $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$

Answer:

$$\text{Basis: For n } = 1\text{: } \sum_{i=1}^{1} i^3 = (\sum_{i=1}^{1} i)^2 \rightarrow LHS = RHS = 1 \; Proved$$

$$\text{Hypothesis: For n } = \text{ k: } \sum_{i=1}^{k} i^3 = (\sum_{i=1}^{k} i)^2$$

$$\text{Proof: For n } = \text{ k} + 1\text{: } \sum_{i=1}^{k+1} i^3 =? (\sum_{i=1}^{k+1} i)^2$$

$$LHS = \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3,$$

$$RHS = \left(\sum_{i=1}^{k} i + k + 1\right)^2 = \left(\sum_{i=1}^{k} i\right)^2 + 2 \times \sum_{i=1}^{k} i \times (k+1) \; + \; (k+1)^2$$

$$As: (k+1)^3 = 2 \times \sum_{i=1}^{k} i \times (k+1) \; + \; (k+1)^2 \; then \; LHS == RHS$$

b: $1 + 2^n < 3^n, for \; all \; n \geq 2, n \in N$

Answer:

$$n = 2: 1 + 2^2 <? \; 3^2 \rightarrow 5 < 9 \; Proved$$

$$\text{Hypothesis: For n } = \text{ k: } 1 + 2^k < 3^k \rightarrow 2^k < 3^k - 1$$

$$\text{Proof: For n } = \text{ k} + 1\text{: } 1 + 2^{k+1} <? \; 3^{k+1}$$

$$1 + 2^k \times 2 <? \; 3^k \times 3 \rightarrow 1 + 2^k \times 2 < 1 + \left(3^k - 1\right) \times 2 <? \; 3^k \times 3$$

$$1 + 2 \times 3^k - 2 <? \; 3^k \times 3 \rightarrow -1 <? \; 3^n \; As \; n \geq 2 \; then \; LHS \leq RHS$$

Q2. Prove that, if $(1+x) > 0$ then, $(1+x)^n \geq 1 + nx, for\ all\ n\ \in\ N$

Answer:

$$n = 1: (1+x)^1 \geq 1 + 1 \times x\ \rightarrow\ Proved$$

$$\text{Hypothesis: For n } = \text{ k}: (1+x)^k \geq 1 + kx$$

$$\text{Proof: For n } = \text{ k} + 1: (1+x)^{k+1} \geq ?\, 1 + (k+1)x$$

$$(1+x)(1+x)^k \geq \big((1+x)(1+kx) = 1 + kx + x + kx^2\big)$$

$$1 + kx + x + kx^2 = 1 + (k+1)x + kx^2$$

$$As\ kx^2 \geq 0\ then\ 1 + (k+1)x + kx^2\ \geq\ 1 + (k+1)x \rightarrow LHS\ \geq RHS$$

Q3. Let $\Sigma = \{a,b\}$, in each of the following three cases, give an example of languages L_1 and L_2 that satisfy the stated conditions. (Note that $L_1 \subseteq \Sigma^*$, and $L_2 \subseteq \Sigma^*$).

I. $L_1L_2 = L_2L_1$, such that $L_1 \subseteq L_2$, $L_2 \not\subset L_1$, and $L_1 \neq \{\varepsilon\}$
II. $(L_1 \cup L_2)^* \neq L_1^* \cup L_2^*$
III. $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$, such that $L_1 \not\subset L_2$, and $L_2 \not\subset L_1$

Answer:

I. $L_1 = \phi$, $L_2 = \{\varepsilon\}$
II. $L_1 = \{a\}$, $L_2 = \{b\}$
III. $L_1 = \{\varepsilon\}$, $L_2 = \{a\}$

Q4. If r is a real number not equal to 1, prove the following formula:

$$\sum_{i=0}^{n} r^i = (1 - r^{n+1}) / (1 - r) \qquad \text{for } n \geq 0$$

Answer:

$$\text{Basis: For n } = \text{ 0}: \sum_{i=1}^{0} r^0 = \frac{(1 - r^1)}{(1 - r)} \rightarrow LHS = RHS = 1\ Proved$$

$$\text{Hypothesis: For n } = \text{ k}: \sum_{i=1}^{k} r^i = \frac{(1 - r^{k+1})}{(1 - r)}$$

$$\text{Proof: For n } = \text{ k} + 1: \sum_{i=1}^{k+1} r^i =?\, \frac{(1 - r^{k+2})}{(1 - r)}$$

$$LHS = \sum_{i=1}^{k} r^i + r^{k+1} = \frac{(1 - r^{k+1})}{(1 - r)} + r^{k+1} = \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{(1 - r)}$$

$$LHS = \frac{1 - r^{k+2}}{(1 - r)} = RHS \rightarrow Proved$$

Q5. In each case give an example of languages L_1 and L_2 that satisfy the stated conditions.

i) $L_1L_2 = L_2L_1$, $L_1 \subseteq L_2$, $L_2 \not\subset L_1$, $L_1 \neq \{\lambda\}$.

ii) $L_1L_2 = L_2L_1$, $L_1 \not\subset L_2$, $L_2 \not\subset L_1$, $L_1 \neq \{\lambda\}$, $L_2 \neq \{\lambda\}$.

iii) $L_1 \subseteq \{a,b\}^*$, $L_2 \subseteq \{a,b\}^*$, $(L_1 \cup L_2)^* \neq L_1^* \cup L_2^*$.

iv) $L_1 \subseteq \{a,b\}^*$, $L_2 \subseteq \{a,b\}^*$, $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$, $L_1 \not\subset L_2$, $L_2 \not\subset L_1$.

Answer:

$i)\ L_1 = \varphi, L_2 = \{\lambda\}$
$ii)\ L_1 = \{a\}, L_2 = \{aa\}$
$iii)\ L_1 = \{a\}, L_2 = \{b\}$
$iv)\ L_1 = \{\lambda\}, L_2 = \{a\}$

CMPE471 – Tutorial 2

Q1. Let $L_1 = \{w \in \{a,b\}^* \mid n_a(w) > n_b(w)\}$ and $L_2 = \{w \in \{a,b\}^* \mid n_a(w) < n_b(w)\}$. Let $L = \{a,b\}^* - (L_1 \cup L_2)^*$. Describe L and justify your answer.

Answer:

Since $a \epsilon L_1$ and $b \epsilon L_2 \Rightarrow (L_1 \cup L_2)^* = \{a,b\}^* \Rightarrow L = \emptyset$.

i.e. $L_1 = \{a, \dots\}, L_2 = \{b, \dots\} \Rightarrow L_1 \cup L_2 = \{a, b, \dots\} \Rightarrow (L_1 \cup L_2)^* = \{a,b\}^*$

Q2. Construct CFGs for the following languages:

i) $L = \{a^j b^k a^n \mid k = j + n\}$

ii) $L = \{a^k b^{2k} c^n \mid k, n > 0\}$

iii) $L = \{a^j b^k c^n \mid 0 \le j + k \le n\}$

iv) $L = \{a,b\}^* - \{a^n b^n \mid n \ge 0\}$

Answer:

i) $S \to AB$
$A \to aAb \mid \varepsilon$
$B \to bBa \mid \varepsilon$

ii) $S \to Sc \mid Ac$
$A \to aAbb \mid abb$

iii) $S \to aSc \mid Sc \mid A$
$A \to bAc \mid \varepsilon$

iv) $S \to A \mid B \mid C$
$A \to aA \mid bA \mid Aa \mid Ab \mid ba$
$B \to aBb \mid Bb \mid b$
$C \to aCb \mid aC \mid a$

Q3. Find the languages generated by the following CFGs:

i) $S \to aSbb \mid aSb \mid aS \mid \varepsilon$

ii) $S \to aScc \mid aAcc$
$A \to bAc \mid bc$

iii) $S \to aSb \mid aSbb \mid aSbbb \mid \varepsilon$

iv) $S \to aSbS \mid bSaS \mid \varepsilon$

v) $S \to aS \mid cS \mid bA \mid \varepsilon$
$A \to aS \mid cS \mid \varepsilon$

vi) $S \to aSbb \mid A$
$A \to cA \mid c$

Answer:

i) $L(G) = \{a^n b^m \mid m \le 2n\}$

ii) $L(G) = \{a^k b^n c^{2k+n} \mid k, n > 0\}$

iii) $L(G) = \{a^n b^m \mid n \le m \le 3n\}$

iv) $L(G) = \{w \in \{a,b\}^* \mid n_a(w) = n_b(w)\}$

v) $L(G) = \{w \in \{a,b,c\}^* \mid w \text{ does not contain } bb\}$

vi) $L(G) = \{a^k c^n b^{2k} \mid k \ge 0, n \ge 1\}$

Q4. Show that the grammar $S \rightarrow aSb \mid bSa \mid SS \mid \varepsilon$ is ambiguous.

Answer: $S \Rightarrow \varepsilon$ $\quad$ $S \Rightarrow SS \Rightarrow S \Rightarrow \varepsilon$

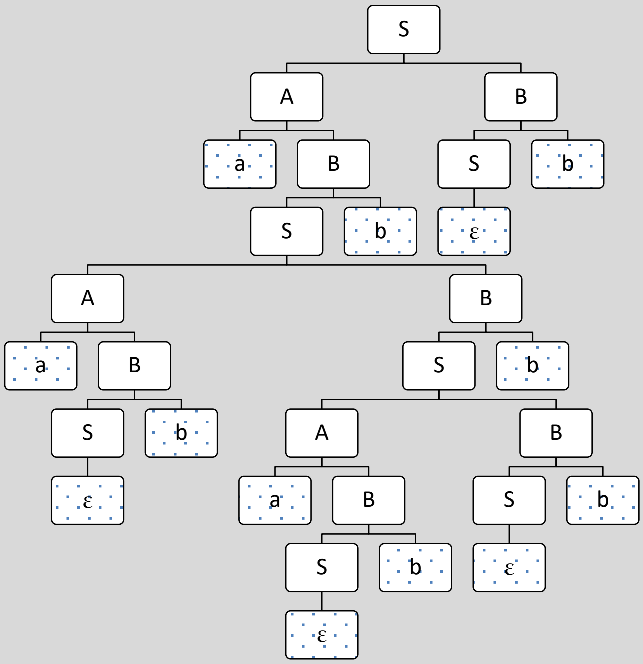
Q5. Consider the CFG with the following products. Find the derivation tree of *aababbbbb*.

$S \rightarrow AB \mid \varepsilon$

$A \rightarrow aB$

$B \rightarrow Sb$

Answer:



CMPE471 – Tutorial 3

Q1. Consider the regular grammar G = ({S,A}, {a,b}, P, S) where P consists of the following productions:

$$G: S \rightarrow bS \mid aA \mid \varepsilon$$
$$A \rightarrow bA \mid aS$$

a) Describe the language generated by this grammar.

b) Give a minimal regular expression for L(G).

c) Find an equivalent grammar to G.

Answer:

vii) $L(G) = \{w \mid w \in \{a,b\}^* \}\ and\ w\ contains\ an\ even\ number\ of\ a's\ \}$

viii) b*(ab* ab*)* or (b*ab*a)*b*

ix) $G: S \rightarrow aA \mid bS \mid \varepsilon$
$A \rightarrow aB \mid bA \mid a$
$B \rightarrow aA \mid bB \mid b$

Q2. Give minimal regular expressions for the following languages:

a) The set of all strings over {a, b, c} that starts and end with the same symbol.

b) The set of all strings over {a, b} in which every pair of adjacent a's appears before any pair of adjacent b's.

c) The set of all strings over {0, 1} except for the two strings 11 and 111.

d) The set of all strings over {0, 1} that have an even number of 0's or exactly three 1's.

e) The set of all strings over {a, b, c} such that every a is followed by at least two c's.

Answer:

a) a (a + b + c)* a + b (a + b + c)* b + c (a + b + c)* c + a +b+ c + ε

b) (b + ε)(a + <u>ab</u>)* (b + ab)*(a + ε)

c) ε + 1 + (0 + 1)* 0 (0 + 1)* + (1111) 1*

d) (1* 01* 01*)* + 0* 10* 10* 10*

e) (acc + b + c)*

Q3. Determine whether each of the following statements is true (T) or false (F). In case of being false write a short comment, or give a counter example.

	Statement	T/F	Comment / Counter Example
1)	a*(ba*)* = (a + b)*	T	
2)	L[a*b*] ∩ L[c*d*] = { }	F	Both languages contain ε ⇒ L[a*b*] ∩ L[c*d*] = { ε }
3)	If L_1 and L_2 are not regular then $L_1 \cup L_2$ is also not regular.	F	Take $L_1 = \{a^n b^m \mid n < m\}$ $L_2 = \{a^n b^m \mid n \geq m\}$ Both are not regular, but $L_1 \cup L_2$ = L[a*b*] which is regular.
4)	If L_1 is regular and $L_1 \cup L_2$ is also regular, then L_2 must be regular.	F	Take L_1 = L[a + b]*, $L_2 = \{a^n b^n \mid n \geq 1\}$ $L_1 \cup L_2 = L_1$, but L_2 is not regular.
5)	The set of even integers is closed under division.	F	e.g. 6/2 = 3

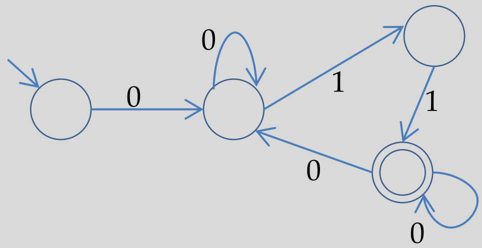
CMPE471 – Tutorial 4

Q1. Consider the set of strings over the alphabet {0,1} obeying the following conditions:

a) The number of 1's in a string is <u>even and at least two.</u>
b) There are <u>no more than two</u> 1's <u>successively</u>.
c) 01 is always <u>followed by</u> 1.
d) The strings always start with 0.

Find a regular expression that denotes the language described by the above set of strings.

Answer: The following finite automation accepts the language:

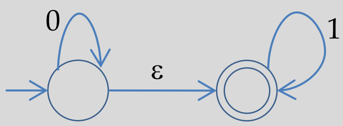


Regular expression: 00* 110* (0* 0110*)*

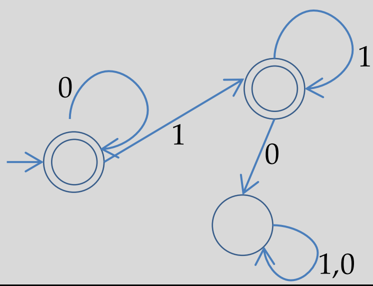
Q2. Find a DFA that accepts the language denoted by 0* 1*.

Answer:

First NFA-ε:



After applying NFA-ε to DFA conversion:



Q3. Describe using set notation the language denoted by aa(a+bϕ)*.

Answer: $\{a^n \mid n > 1\}$

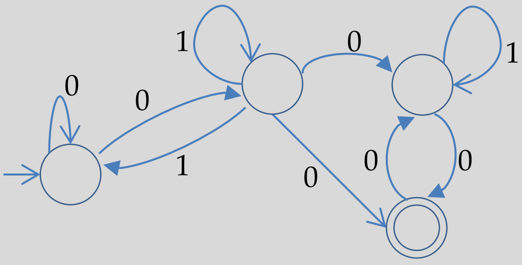
Q4. Find a Deterministic Finite Automaton that accepts the language generated by the following grammar.

$S \rightarrow 0S \mid 0A$
$A \rightarrow 0B \mid 0C \mid 1S \mid 1A \mid 0$
$B \rightarrow 0C$

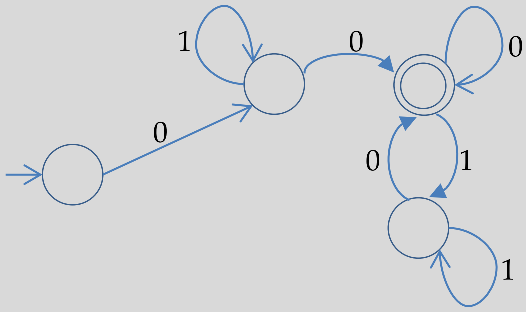
C → 0B | 1C | 0

Answer:

First NFA:



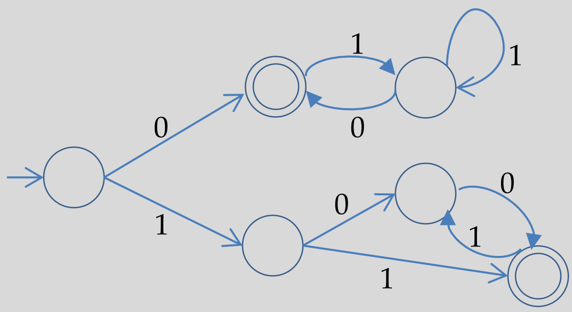
Then DFA:



Q5. Give the language generated by the following grammar as a <u>regular expression</u>.

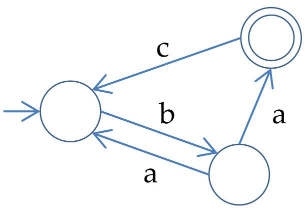
S → 0A | 1C | 0
A → 1B
B → 0A | 1B | 0
C → 0D | 1E | 1
D → 0E | 0
E → 1D

Answer:



0(11* 0)* + 1 [00 (10)* + 1(10)*] = 0 (11* 0)* + 1 (00 + 1)(10)*

Q6. We are given the following non-deterministic finite automation M. Find a deterministic automation D that accepts the same language as M.



Answer: