The Karnaugh Map

Boolean Function Minimization

- Complexity of a Boolean function is directly related to the complexity of the algebraic expression
- The truth table of a function is unique
- However, the algebraic expression is not unique
- Boolean function can be simplified by algebraic manipulation
- However, algebraic manipulation depends on experience
- Algebraic manipulation does not guarantee that the simplified Boolean expression is minimal

Example: Sum of Minterms

Truth Table

x y z	f	Minterm
0 0 0	0	
0 0 1	1	$m_1 = x'y'z$
0 1 0	1	$m_2 = x'yz'$
0 1 1	1	$m_3 = x'yz$
1 0 0	0	
1 0 1	1	$m_5 = xy'z$
1 1 0	0	
1 1 1	1	$m_7 = xyz$

Focus on the '**1**' entries

$$f = m_1 + m_2 + m_3 + m_5 + m_7$$

$$f = \sum(1, 2, 3, 5, 7)$$

$$f = x'y'z + x'yz' + x'yz + xy'z + xyz$$

❖ Sum-of-Minterms has 15 literals ➔ Can be simplified

Algebraic Manipulation

❖ **Simplify:** $f = x'y'z + x'yz' + x'yz + xy'z + xyz$ (15 literals)

$f = x'y'z + x'yz' + x'yz + xy'z + xyz$	(Sum-of-Minterms)
$f = x'y'z + x'yz + x'yz' + xy'z + xyz$	Reorder
$f = x'z(y' + y) + x'yz' + xz(y' + y)$	Distributive · over +
$f = x'z + x'yz' + xz$	Simplify (7 literals)
$f = x'z + xz + x'yz'$	Reorder
$f = (x' + x)z + x'yz'$	Distributive · over +
$f = z + x'yz'$	Simplify (4 literals)
$f = (z + x'y)(z + z')$	Distributive + over ·
$f = z + x'y$	Simplify (3 literals)

Drawback of Algebraic Manipulation

- No clear steps in the manipulation process
  - Not clear which terms should be grouped together
  - Not clear which property of Boolean algebra should be used next
- Does not always guarantee a minimal expression
  - Simplified expression may or may not be minimal
  - Different steps might lead to different non-minimal expressions
- However, the goal is to minimize a Boolean function
- Minimize the **number of literals** in the Boolean expression
  - The **literal count** is a good measure of the **cost** of logic implementation
  - Proportional to the number of transistors in the circuit implementation

Karnaugh Map

- Called also K-map for short
- The Karnaugh map is a diagram made up of squares
- It is a reorganized version of the truth table
- Each square in the Karnaugh map represents a minterm
- Adjacent squares differ in the value of one variable
- Simplified expressions can be derived from the Karnaugh map
  - By recognizing patterns of squares
- Simplified sum-of-products expression (AND-OR circuits)
- Simplified product-of-sums expression (OR-AND circuits)

Two-Variable Karnaugh Map

- Minterms m_0 and m_1 are adjacent (also, m_2 and m_3)
  - They differ in the value of variable y
- Minterms m_0 and m_2 are adjacent (also, m_1 and m_3)
  - They differ in the value of variable x

Two-variable K-map

$x \backslash y$	0	1
0	m_0	m_1
1	m_2	m_3

$x \backslash y$	0	1
0	$x'y'$	$x'y$
1	xy'	xy

From a Truth Table to Karnaugh Map

- Given a truth table, construct the corresponding K-map
- Copy the function values from the truth table into the K-map
- Make sure to copy each value into the proper K-map square

Truth Table

x y	f
0 0	1
0 1	0
1 0	1
1 1	1

K-map

$x \backslash y$	0	1
0	1	0
1	1	1

K-Map Function Minimization

- Two adjacent cells containing 1's can be combined
- $f = m_0 + m_2 + m_3$
- $f = x'y' + xy' + xy$ (6 literals)
- $m_0 + m_2 = x'y' + xy' = (x' + x)y' = y'$
- $m_2 + m_3 = xy' + xy = x(y' + y) = x$
- Therefore, f can be simplified as: $f = x + y'$ (2 literals)

K-map

$x \backslash y$	0	1
0	1	0
1	1	1

Three-Variable Karnaugh Map

- Have eight squares (for the 8 minterms), numbered 0 to 7
- The last two columns are not in numeric order: 11, 10
  - Remember the numbering of the squares in the K-map
- Each square is adjacent to three other squares
- Minterms in adjacent squares can always be combined
  - This is the key idea that makes the K-map work
- Labeling of rows and columns is also useful

$x \backslash yz$	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

	$x \backslash yz$	00 (y')	01 (y')	11 (y)	10 (y)
x'	0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
x	1	$xy'z'$	$xy'z$	xyz	xyz'
		z'	z	z	z'

Simplifying a Three-Variable Function

Simplify the Boolean function: $f(x, y, z) = \sum(3, 4, 5, 7)$

$f = x'yz + xy'z' + xy'z + xyz$ (12 literals)

1. Mark **'1'** all the K-map squares that represent function f

2. Find possible adjacent squares

$x'yz + xyz = (x' + x)yz = yz$

$xy'z' + xy'z = xy'(z' + z) = xy'$

Therefore, $f = xy' + yz$ (4 literals)

$x \backslash yz$	00	01	11	10
	y'		y	
x' 0	0	0	1	0
x 1	1	1	1	0
	z'	z		z'

Simplifying a Three-Variable Function (2)

Here is a second example: $f(x, y, z) = \sum(3, 4, 6, 7)$

$f = x'yz + xy'z' + xyz' + xyz$ (12 literals)

Learn the locations of the 8 indices based on the variable order

$x'yz + xyz = (x' + x)yz = yz$

Corner squares can be combined

$xy'z' + xyz' = xz'(y' + y) = xz'$

Therefore, $f = xz' + yz$ (4 literals)

$x \backslash yz$	00	01	11	10
	y'		y	
x' 0	0	0	1	0
x 1	1	0	1	1
	z'	z		z'

Combining Squares on a 3-Variable K-Map

- By combining squares, we reduce number of literals in a product term, thereby reducing the cost
- On a 3-variable K-Map:
  - One square represents a minterm with 3 variables
  - Two adjacent squares represent a term with 2 variables
  - Four adjacent squares represent a term with 1 variable
  - Eight adjacent square is the constant **‘1’** (no variables)

Example of Combining Squares

- Consider the Boolean function: $f(x, y, z) = \sum(2, 3, 5, 6, 7)$
- $f = x'yz' + x'yz + xy'z + xyz' + xyz$
- The four minterms that form the 2×2 red square are reduced to the term y
- The two minterms that form the blue rectangle are reduced to the term xz
- Therefore: $f = y + xz$

x \ yz	00 (y')	01 (y')	11 (y)	10 (y)
x' 0	0	0	1	1
x 1	0	1	1	1
	z'	z	z	z'

$$x'yz + x'yz' + xyz + xyz'$$
$$= x'y(z + z') + xy(z + z')$$
$$= x'y + xy = (x' + x)y = y$$

Minimal Sum-of-Products Expression

Consider the function: $f(x, y, z) = \sum(0, 1, 2, 4, 6, 7)$

Find a minimal sum-of-products (SOP) expression

Solution:

Red block: term = z'

Green block: term = $x'y'$

Blue block: term = xy

	yz	y' 00	01	11	y 10
x'	x 0	1	1	0	1
x	1	1	0	1	1
		z'	z	z	z'

Minimal sum-of-products: $f = z' + x'y' + xy$ (5 literals)

Four-Variable Karnaugh Map

4 variables ➔ 16 squares

Remember the numbering of the squares in the K-map

Each square is adjacent to four other squares

Notice the order of Rows 11 and 10 and the order of columns 11 and 10

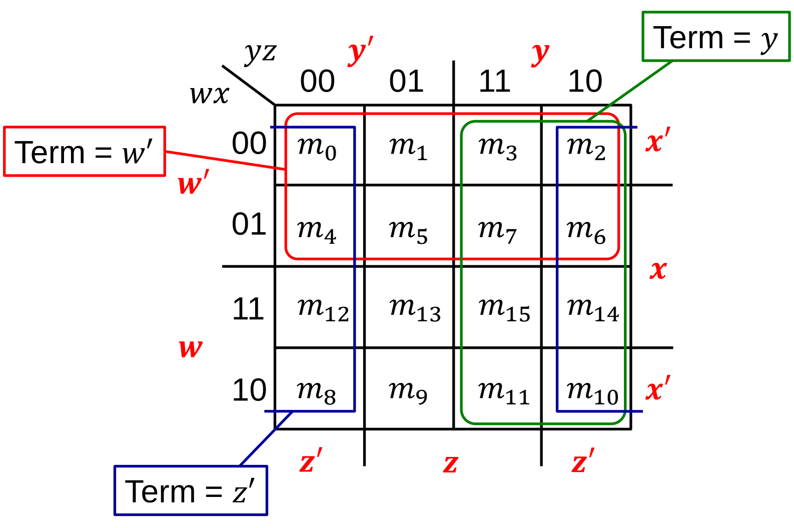
$m_0 = w'x'y'z'$ $m_1 = w'x'y'z$
$m_2 = w'x'y\,z'$ $m_3 = w'x'y\,z$
$m_4 = w'x\,y'z'$ $m_5 = w'x\,y'z$
$m_6 = w'x\,y\,z'$ $m_7 = w'x\,y\,z$
$m_8 = w\,x'y'z'$ $m_9 = w\,x'y'z$
$m_{10} = w\,x'yz'$ $m_{11} = w\,x'y\,z$
$m_{12} = w\,x\,y'z'$ $m_{13} = w\,x\,y'z$
$m_{14} = w\,x\,y\,z'$ $m_{15} = w\,x\,y\,z$

	wx \ yz	y' 00	01	11	y 10	
w'	00	m_0	m_1	m_3	m_2	x'
w'	01	m_4	m_5	m_7	m_6	x
w	11	m_{12}	m_{13}	m_{15}	m_{14}	x
w	10	m_8	m_9	m_{11}	m_{10}	x'
		z'	z	z	z'	

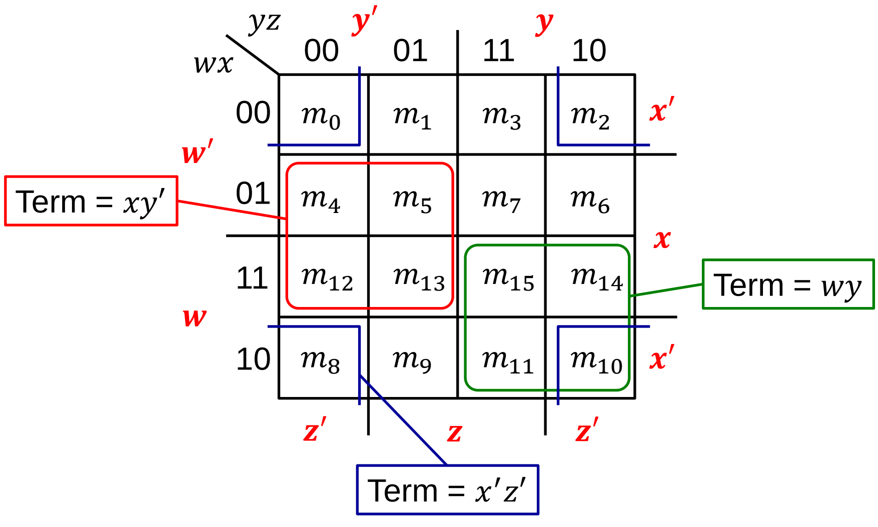
Combining Squares on a 4-Variable K-Map

- On a 4-variable K-Map:
  - One square represents a minterm with 4 variables
  - Two adjacent squares represent a term with 3 variables
  - Four adjacent squares represent a term with 2 variables
  - Eight adjacent squares represent a term with 1 variable
  - Combining all 16 squares is the constant **'1'** (no variables)

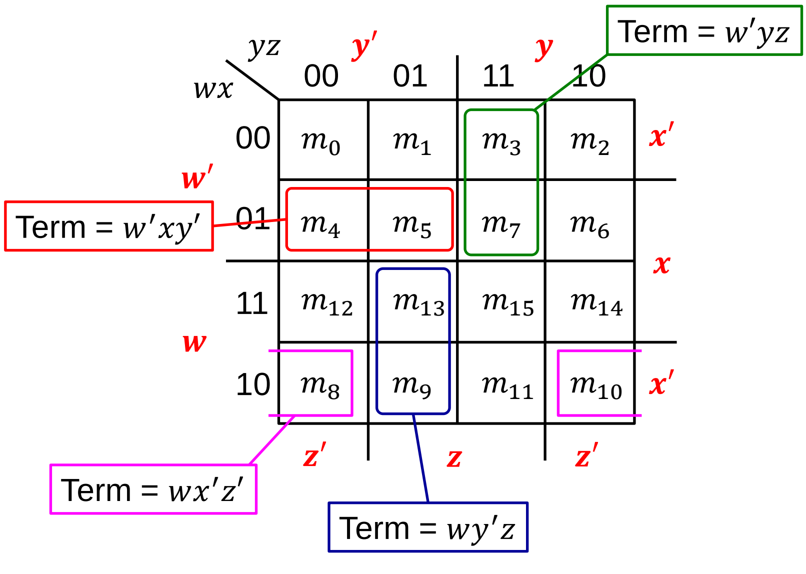
Combining Eight Squares



Combining Four Squares



Combining Two Squares



Simplifying a 4-Variable Function

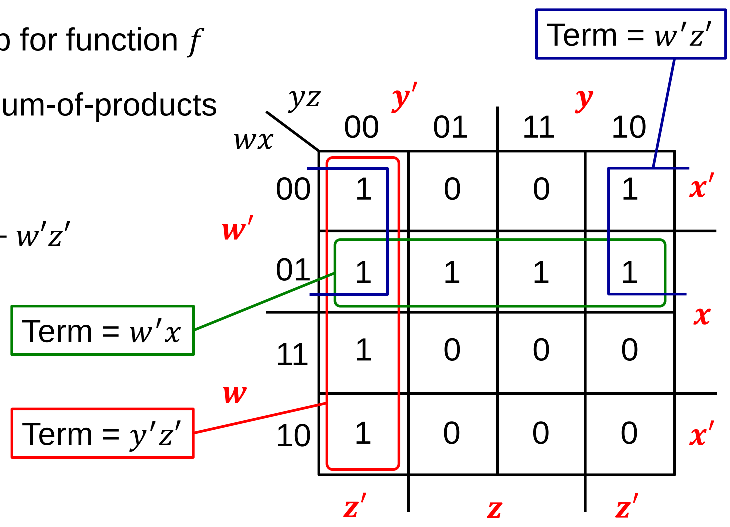
Given $f(w,x,y,z) = \sum(0,2,4,5,6,7,8,12)$

Draw the K-map for function f

Minimize f as sum-of-products

Solution:

$f = w'x + y'z' + w'z'$



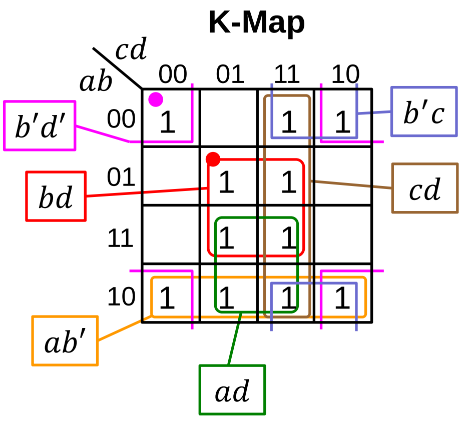
Prime Implicants

- **Prime Implicant:** a product term obtained by combining the **maximum number of adjacent squares** in the K-map
- The number of combined squares must be a **power of 2**
- **Essential Prime Implicant:** is a prime implicant that covers at least one minterm not covered by the other prime implicants
- The prime implicants and essential prime implicants can be determined by inspecting the K-map

Example of Prime Implicants

Find all the prime implicants and essential prime implicants for:

$f(a,b,c,d) = \sum(0,2,3,5,7,8,9,10,11,13,15)$



Six Prime Implicants

bd, $b'd'$, ab', ad, cd, $b'c$

Only Two Prime Implicants are essential

bd and $b'd'$

Simplification Procedure Using the K-Map

1. Find all the essential prime implicants
   - ✧ Covering maximum number (power of 2) of 1's in the K-map
   - ✧ Mark the minterm(s) that make the prime implicants essential
2. Add prime implicants to cover the function
   - ✧ Choose a minimal subset of prime implicants that cover all remaining 1's
   - ✧ Make sure to cover all 1's not covered by the essential prime implicants
   - ✧ Minimize the overlap among the additional prime implicants

❖ Sometimes, a function has multiple simplified expressions

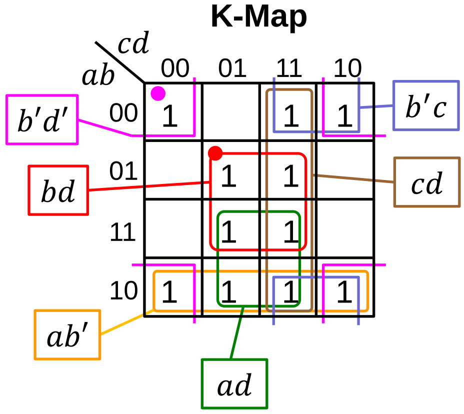
- ✧ You may be asked to list all the simplified sum-of-product expressions

Obtaining All Minimal SOP Expressions

Consider again: $f(a,b,c,d) = \sum(0,2,3,5,7,8,9,10,11,13,15)$

Obtain all minimal sum-of-products (SOP) expressions

K-Map



Two essential Prime Implicants: bd and $b'd'$

Four possible solutions:

$f = bd + b'd' + cd + ad$

$f = bd + b'd' + cd + ab'$

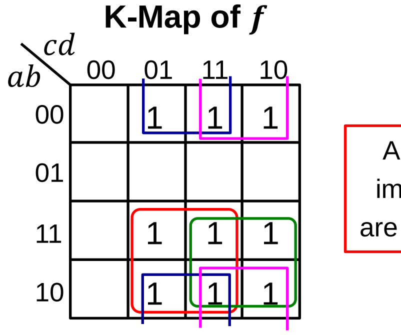
$f = bd + b'd' + b'c + ab'$

$f = bd + b'd' + b'c + ad$

Product-of-Sums (POS) Simplification

- All previous examples were expressed in Sum-of-Products form
- With a minor modification, the Product-of-Sums can be obtained
- Example: $f(a,b,c,d) = \sum(1,2,3,9,10,11,13,14,15)$

K-Map of f

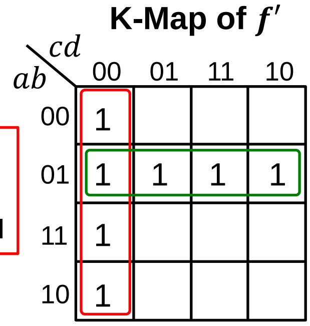


$f = ad + ac + b'd + b'c$

Minimal Sum-of-Products = 8 literals

All prime implicants are essential

K-Map of f'



$f' = c'd' + a'b$

$f = (c+d)(a+b')$

Minimal Product-of-Sums = 4 literals

Product-of-Sums Simplification Procedure

1. Draw the K-map for the function f
   - ✧ Obtain a minimal Sum-of-Products (SOP) expression for f
2. Draw the K-map for f', replacing the 0's of f with 1's in f'
3. Obtain a minimal Sum-of-Products (SOP) expression for f'
4. Use DeMorgan's theorem to obtain $f = (f')'$
   - ✧ The result is a minimal Product-of-Sums (POS) expression for f
5. Compare the cost of the minimal SOP and POS expressions
   - ✧ Count the number of literals to find which expression is minimal

Don't Cares

- ❖ Sometimes, a function table may contain entries for which:
  - ✧ The input values of the variables will never occur, or
  - ✧ The output value of the function is never used
- ❖ In this case, the output value of the function is not defined
- ❖ The output value of the function is called a **don't care**
- ❖ A don't care is an **X** value that appears in the function table
- ❖ The **X** value can be later chosen to be **0 or 1**
  - ✧ To minimize the function implementation

Example of a Function with Don't Cares

- Consider a function f defined over BCD inputs
- The function input is a BCD digit from 0 to 9
- The function output is 0 if the BCD input is 0 to 4
- The function output is 1 if the BCD input is 5 to 9
- The function output is X (don't care) if the input is 10 to 15 (not BCD)
- $f = \underbrace{\sum_m (5,6,7,8,9)}_{\text{Minterms}} + \underbrace{\sum_d (10,11,12,13,14,15)}_{\text{Don't Cares}}$

Truth Table

a b c d	f
0 0 0 0	0
0 0 0 1	0
0 0 1 0	0
0 0 1 1	0
0 1 0 0	0
0 1 0 1	1
0 1 1 0	1
0 1 1 1	1
1 0 0 0	1
1 0 0 1	1
1 0 1 0	X
1 0 1 1	X
1 1 0 0	X
1 1 0 1	X
1 1 1 0	X
1 1 1 1	X

Minimizing Functions with Don't Cares

Consider: $f = \sum_m (5,6,7,8,9) + \sum_d (10,11,12,13,14,15)$

If the don't cares were treated as 0's we get:

$f = a'bd + a'bc + ab'c'$ (9 literals)

If the don't cares were treated as 1's we get:

$f = a + bd + bc$ (5 literals)

The don't care values can be selected to be either 0 or 1, to produce a minimal expression

K-Map of f

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	0	1	1	1
11	X	X	X	X
10	1	1	X	X

Simplification Procedure with Don't Cares

1. Find all the essential prime implicants
   - ✧ Covering maximum number (power of 2) of 1's and X's (don't cares)
   - ✧ Mark the 1's that make the prime implicants essential
2. Add prime implicants to cover the function
   - ✧ Choose a minimal subset of prime implicants that cover all remaining 1's
   - ✧ Make sure to cover all 1's not covered by the essential prime implicants
   - ✧ Minimize the overlap among the additional prime implicants
   - ✧ You need not cover all the don't cares (some can remain uncovered)

❖ Sometimes, a function has multiple simplified expressions

Minimizing Functions with Don't Cares (2)

Simplify: $g = \sum_m(1,3,7,11,15) + \sum_d(0,2,5)$

Solution 1: $g = cd + a'b'$ (4 literals)

Solution 2: $g = cd + a'd$ (4 literals)

Prime Implicant cd is essential

K-Map of g

$ab \backslash cd$	00	01	11	10
00	X	1	1	X
01	0	X	1	0
11	0	0	1	0
10	0	0	1	0

K-Map of g

$ab \backslash cd$	00	01	11	10
00	X	1	1	X
01	0	X	1	0
11	0	0	1	0
10	0	0	1	0

Not all don't cares need be covered

Minimal Product-of-Sums with Don't Cares

Simplify: $g = \sum_m(1, 3, 7, 11, 15) + \sum_d (0, 2, 5)$

Obtain a product-of-sums minimal expression

Solution: $g' = \sum_m(4, 6, 8, 9, 10, 12, 13, 14) + \sum_d (0, 2, 5)$

Minimal $g' = d' + ac'$ (3 literals)

Minimal product-of-sums:

$g = d(a' + c)$ (3 literals)

The minimal sum-of-products expression for g had 4 literals

K-Map of g'

$ab \backslash cd$	00	01	11	10
00	X	0	0	X
01	1	X	0	1
11	1	1	0	1
10	1	1	0	1