MIPS arithmetic

- Unsigned and signed number representations.
- Addition and subtraction with two's complement numbers.
- Overflow detection.

ƒ - review the construction of an ALU that will appear in our CPU Design

Unsigned numbers

ƒ We can store unsigned numbers as their binary equivalents.
ƒ Bit position *i* has a decimal weight 2^i, so the decimal value v of an *n*-bit unsigned binary number $a_{n-1}a_{n-2}...a_1a_0$ can be calculated as below.

$$v = \sum_{i=0}^{n-1} 2^i a_i \cdot$$

ƒ This is just the sum of each digit times the weight of its position.

$1\ 0\ 1\ 1\ 0\ 1_2 = (2^5 \bullet 1) + (2^4 \bullet 0) + (2^3 \bullet 1) + (2^2 \bullet 1) + (2^1 \bullet 0) + (2^0 \bullet 1) = 45_{10}$

ƒ The smallest and largest possible numbers are 0 and 2^n-1.

ƒ In MIPS, the largest unsigned number is $2^{32}-1$, or about 4.3 billion.

Hexadecimal notation

- Hexadecimal is frequently used as a shorthand for binary numbers, because one hex digit corresponds to four bits.
- Converting between binary and hex is easy with a table.

$$B\ E\ E\ F_{16} = 1011\ 1110\ 1110\ 1111_2$$

- In SPIM and many high-level programming languages, the prefix 0x denotes a hexadecimal constant, as in 0xBEEF.

Binary	Hex
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	A
1011	B
1100	C
1101	D
1110	E
1111	F

Working with unsigned numbers

- Addition can be done just like in decimal.

```
    1 1 1 1 1
    1 0 1 1 0 1        45
+   0 1 0 1 1 1    +   23
-----------------  ------
  1 0 0 0 1 0 0        68
```

- Multiplication and integer division by powers of 2 can be done using left and right "logical" shifts.

$101101 \times 10 = 1011010$ $\quad 45 \times 2 = 90$

$101101 / 10 = 010110$ $\quad 45 / 2 = 22$

Logical shifts in MIPS

ƒ MIPS has sll (shift left logical) and srl (shift right logical) instructions.
– A constant specifies the number of bits to shift, from 0 to 31.
– 0s are shifted into the right or left sides, respectively.
ƒ For example, assume that $t0 contains 0xFEEDBEEF.

```
sll  $t1, $t0, 4      # $t1 contains 0xEEDBEEF0
srl  $t2, $t0, 12     # $t2 contains 0x000FEEDB
```

ƒ Shifts are actually R-type instructions, not I-type!

op	rs	rt	rd	shamt	func
6 bits	5 bits	5 bits	5 bits	5 bits	6 bits

ƒ The constant is stored in the 5-bit "shamt" field that we hadn't seen up until now.

Unsigned overflow

ƒ One recurring issue in computer arithmetic is dealing with finite amounts of storage, such as 32-bit MIPS registers.
ƒ Overflow occurs when the result of an operation is too large to be stored.
ƒ There are many examples of unsigned *n*-bit overflows.
– When there is a carry out of position *n*–1 in an addition.

ƒ The 6-bit addition and multiplication operations on page 4 both result in overflow, since the correct answers require 7 bits.

Signed two's-complement numbers

- Signed numbers are represented in two's complement format.
- The most significant bit a_{n-1} of each number is a sign bit.
  - 0 indicates a positive number.
  - 1 indicates a negative number.
  - The range of n-bit signed numbers is from -2^{n-1} to $+2^{n-1}-1$
  - For 32-bit values, the range is roughly ±2.15 billion.
  - The ranges are not exactly even since there are 2^{n-1} negative numbers, but just $2^{n-1}-1$ positive numbers.

Negating a two's complement number

- There are three main ways to negate two's complement numbers. As an example, let's consider the six-bit value 101101 (–19).

1. Complement all the bits in the number and then add 1.
   - Complementing the bits in 101101 yields 010010.
   - Adding 1 results in 010011 (+19).
2. Complement the bits to the left of the rightmost 1.
   - The rightmost 1 in 101101 is in position 0.
   - Complementing the bits to its left gives 010011 (+19).
3. Subtract the number from 2^n.
   - 2^6 is 1000000.
   - Then, 1000000 – 101101 = 010011 (+19).

Two's-complement numbers with *n*=4

Decimal	2C
-8	1000
-7	1001
-6	1010
-5	1011
-4	1100
-3	1101
-2	1110
-1	1111
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111

2C	Decimal
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

Working with signed numbers

- Addition with two's complement is the same as with unsigned numbers.
  - The sign bits are included in the addition.
  - The carry out is ignored.

```
    1  1  1  1  1                      -19
    1  0  1  1  0  1
+   0  1  0  1  1  1              +  +  23
-------------------              -------
1   0  0  0  1  0  0                   4
```

- A subtraction operation A - B is treated as an addition of A + (-B).

Signed overflow

ƒ It's also possible for two's-complement operations to overflow.

ƒ Overflow occurs only when we add two numbers of the *same* sign and get a *different* sign for the result.

- Adding two positive numbers should result in a positive number.
- Adding two negative numbers should yield a negative result.

```
  1 0 1 1 1
  0 1 0 0 1 1      +19
+ 0 1 0 1 1 1    + +23
-----------------------
0 1 0 1 0 1 0      - 22

  0 1 0 0 1
  1 0 1 1 0 1      - 19
+ 1 0 1 0 0 1    + - 23
-----------------------
1 0 1 0 1 1 0      + 22
```

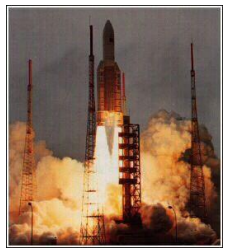

Ariane 5

ƒ In 1996, the European Space Agency's Ariane 5 rocket was launched for the first time... and it exploded 40 seconds after liftoff.

ƒ It turns out the Ariane 5 used software designed for the older Ariane 4.

- The Ariane 4 stored its horizontal velocity as a 16-bit signed integer.
- But the Ariane 5 reaches a much higher velocity, which caused an overflow in the 16-bit quantity.

ƒ The overflow error was never caught, so incorrect instructions were sent to the rocket boosters and main engine.



Sign extension

- We often work with signed numbers of different lengths.
  - Instructions like lw $t0, -4($sp) add 16-bit and 32-bit values together.
  - slti $t0, $t0, 256 compares a 16-bit constant to a 32-bit number. The
  - lb instruction copies an 8-bit value into a 32-bit register.
- You can sign extend a two's complement number by copying the sign bit.
- For instance, we can sign extend 6-bit numbers to 8 bits.

(+23) 010111 ⟶ 00010111 (+23)

(-19) 101101 ⟶ 11101101 (-19)

- If you add 0s instead of sign extending, you may accidentally change the sign, and hence the value, of the result.

(-19) 101101 ⟶ 00101101 (+45)

Interpreting bit patterns

- One of the most important things to remember is that bit patterns have different meanings under different representations!
  - As a six-bit *unsigned* number, 101101 denotes 45 in decimal.
  - But as a *two's complement* number, 101101 denotes -19.
- The example C program below prints the same data (0xFFFFFFFF) twice, but under different interpretations.

```
main()                      // CSIL Sun machines
{
int x = 0xFFFFFFFF;       // 32-bit integers
printf("x = %d\n", x);    // Signed; prints -1
printf("x = %u\n", x);    // Unsigned; prints 4294967295
}
```

Signed and unsigned comparisons

ƒ Specifying the interpretation of binary numbers is especially important in untyped assembly language programming.

ƒ For example, how does 111111 compare to 000000?

– As an unsigned number, 111111 = 63, which is *greater* than 0.

– As a two's complement number, 111111 = -1, which is *less* than 0.

ƒ MIPS includes two versions of the comparison instructions.

– The slt and slti instructions that we saw do signed comparisons.

– The instructions sltu and sltiu perform unsigned comparisons.

```
main()
{
  unsigned int u = 0xFFFFFFFF;
  int s = 0xFFFFFFFF;

  printf("%u is %s than 0\n", u, (u < 0 ? "less" : "greater"));
  printf("%d is %s than 0\n", s, (s < 0 ? "less" : "greater"));
}
```


Signed and unsigned addition

ƒ MIPS provides two versions of the main arithmetic instructions.

– The add, addi and sub instructions will raise an exception if they cause an overflow.

– But addu, addiu, and subu do not raise exceptions on overflow

ƒ Be careful! Both addi and addiu sign extend their constant field, even though addiu is considered an "unsigned" operation.

ƒ In C, overflow exceptions may or may not be raised, depending on the underlying hardware.

```
main()                          // CSIL Sun machines
{
int x = 0x7FFFFFFF;            // Largest 32-bit number
printf("x = %d\n", x);         // Prints  2147483647
x++;
printf("x = %d\n", x);         // Prints -2147483648
}
```

When to use unsigned numbers

ƒ Memory addresses should be treated as unsigned numbers.

– The fourth element of an array that starts at 0x7FFFFFF0 is at address 0x80000000. This address computation should not overflow!

– In fact, stack pushes and pops should be performed using the unsigned add and subtract instructions like addiu, and not addi.

ƒ Non-numeric data should also be handled using unsigned operations.

–

ƒ

MIPS arithmetic instruction summary

ƒ Addition and subtraction

- add, addi, and sub raise exceptions on overflow.
- addu, addiu, and subu do not raise exceptions on overflow.
- addi and addiu both sign extend their immediate fields.

ƒ Comparisons

- slt and slti are for signed comparisons.
- sltu and sltiu do unsigned comparisons.
- slti and sltiu both sign extend their immediate fields.

ƒ Data transfers

– lb sign extends the byte that is loaded into a register.

– lbu does not sign extend the byte.

ƒ Shifting

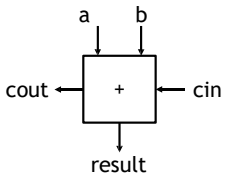
– sll and srl are used for logical shifts of up to 31 bits.

ƒ Bitwise operators

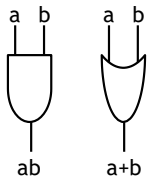
– andi and ori do not sign extend their immediate field.

Hardware for single-bit operations

- We can use a full adder to add data inputs a, b and a carry in cin. This produces a one-bit result and a carry out cout.



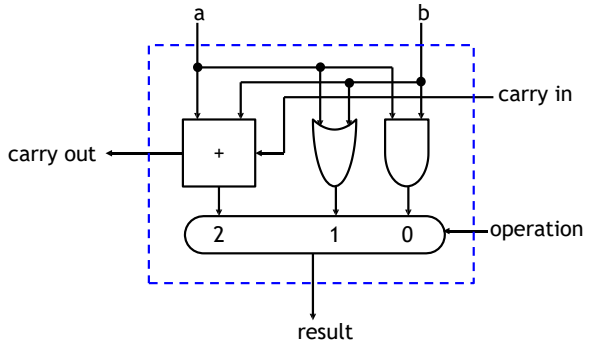
- It's also easy to do one bit and and or operations.



A one-bit ALU

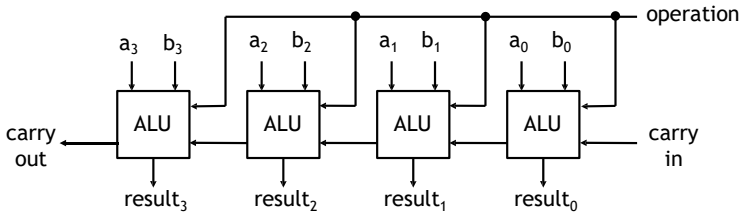
- A simple one-bit ALU can support all three of these functions.
- A two-bit input called operation selects the desired function using a 3-to-1 multiplexer.

operation	result
0	a and b
1	a or b
2	a + b + carry in



An *n*-bit ALU

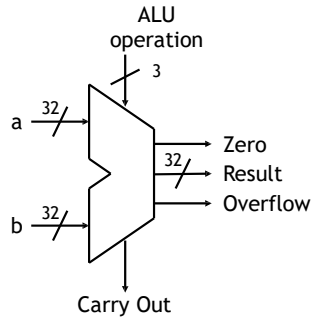
- This one-bit ALU can be replicated to produce an *n*-bit ALU.
  - The carry out from stage *i* connects to the carry in of stage *i*+1.
  - The operation input is shared by all of the one-bit circuits.



A whole MIPS ALU

- We can do the subtraction of a – b by complementing the b input, setting carry in to 1, and performing an addition.
- The ALU also has a slt function, which sets result to 1 or 0 depending on whether or not a is less than b. This can be determined via subtraction.
- Finally, outputs can be generated to signal overflow or a result of zero.

ALU operation	Result
000	a and b
001	a or b
010	a + b
110	a – b
111	a < b



MIPS ALU design

- ALUs are a good example of modular hardware design.
  - 32 full adders can be connected together to form a 32-bit adder.
  - Bitwise operations can also be handled one bit at a time. Adders
  - can also be used to perform subtraction.
- Shift operations are usually handled outside the ALU by a combinational "barrel shifter" circuit, which can shift an arbitrary number of positions in constant time.

Summary

- Signed numbers are represented using the two's complement format.
  - A sign bit distinguishes between positive and negative numbers.
  - Numbers are negated by complementing their bits and adding 1.
  - This makes subtraction hardware easy to build.
- Signed and unsigned numbers have many important differences.
  - Some operations require that we sign extend the operands.
  - Overflow detection differs for signed and unsigned operations.
- The MIPS instruction set includes both signed and unsigned operations, which affect how constant operands and overflows are handled.