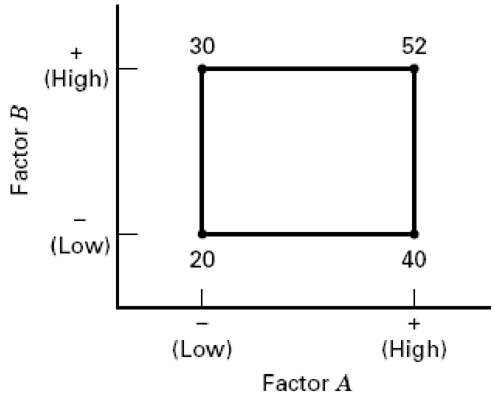
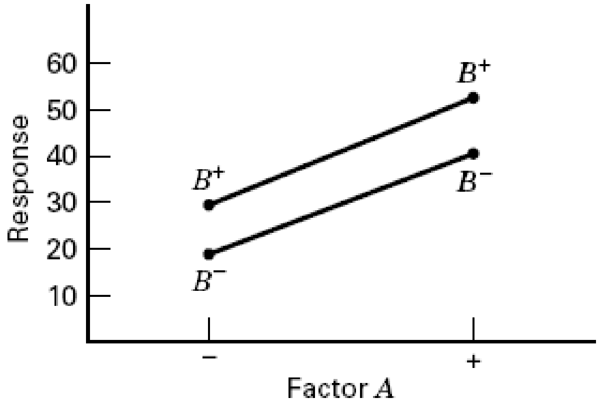
Design of Engineering Experiments – Introduction to Factorials

- Text reference, Chapter 5
- **General principles** of factorial experiments
- The **two-factor factorial** with fixed effects
- The **ANOVA** for factorials
- Extensions to more than two factors

Some Basic Definitions



■ FIGURE 5.1 A two-factor factorial experiment, with the response (y) shown at the corners



■ FIGURE 5.3 A factorial experiment without interaction

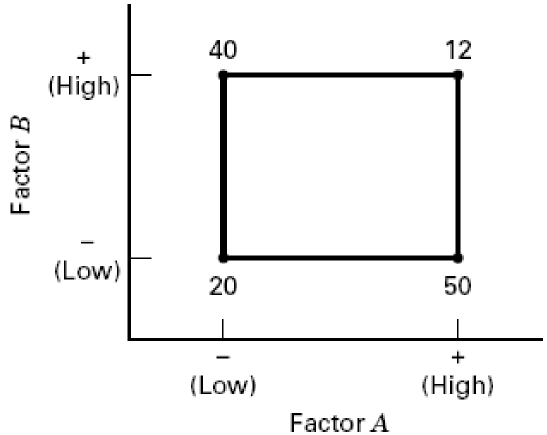
Definition of a factor effect: The change in the mean response when the factor is changed from low to high

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{40+52}{2} - \frac{20+30}{2} = 21$$

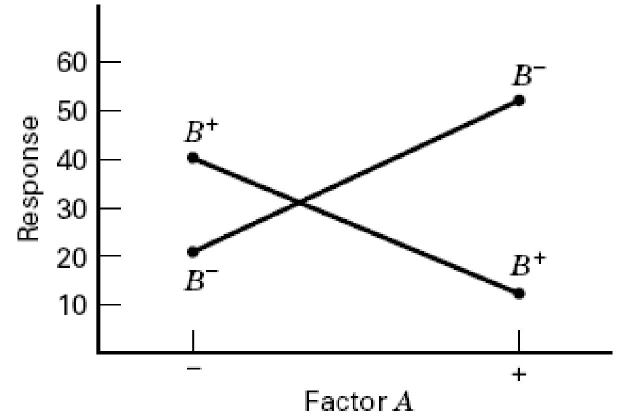
$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{30+52}{2} - \frac{20+40}{2} = 11$$

$$AB = \frac{52+20}{2} - \frac{30+40}{2} = -1$$

The Case of Interaction:



■ **FIGURE 5.2** A two-factor factorial experiment with interaction



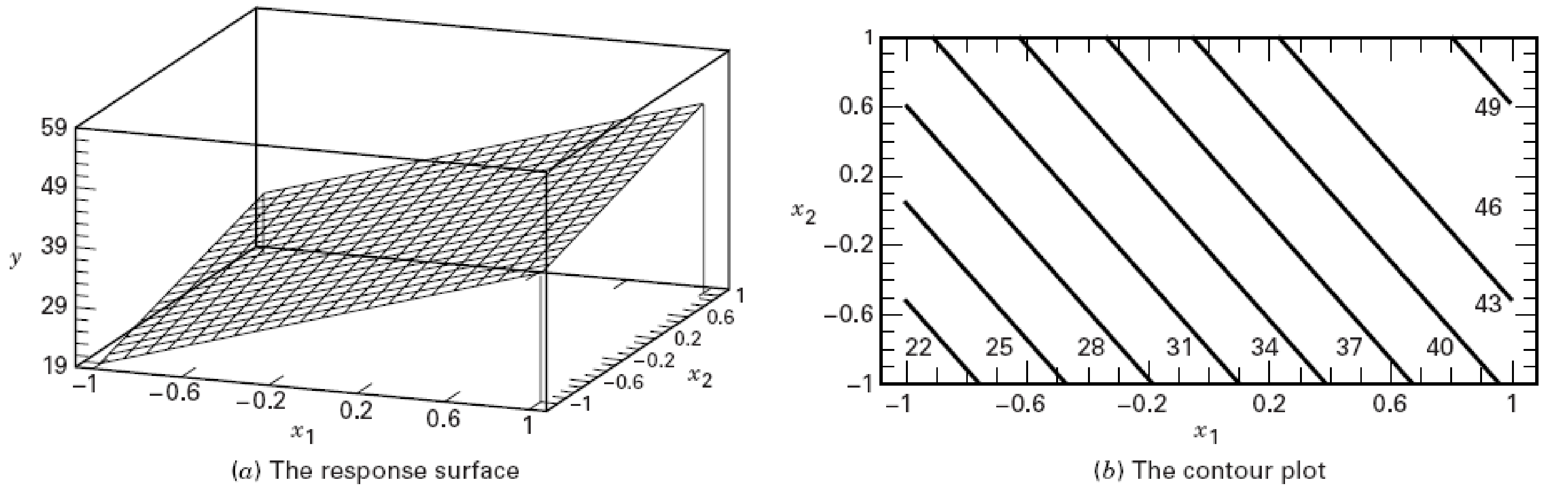
■ **FIGURE 5.4** A factorial experiment with interaction

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{50+12}{2} - \frac{20+40}{2} = 1$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{40+12}{2} - \frac{20+50}{2} = -9$$

$$AB = \frac{12+20}{2} - \frac{40+50}{2} = -29$$

Regression Model & The Associated Response Surface



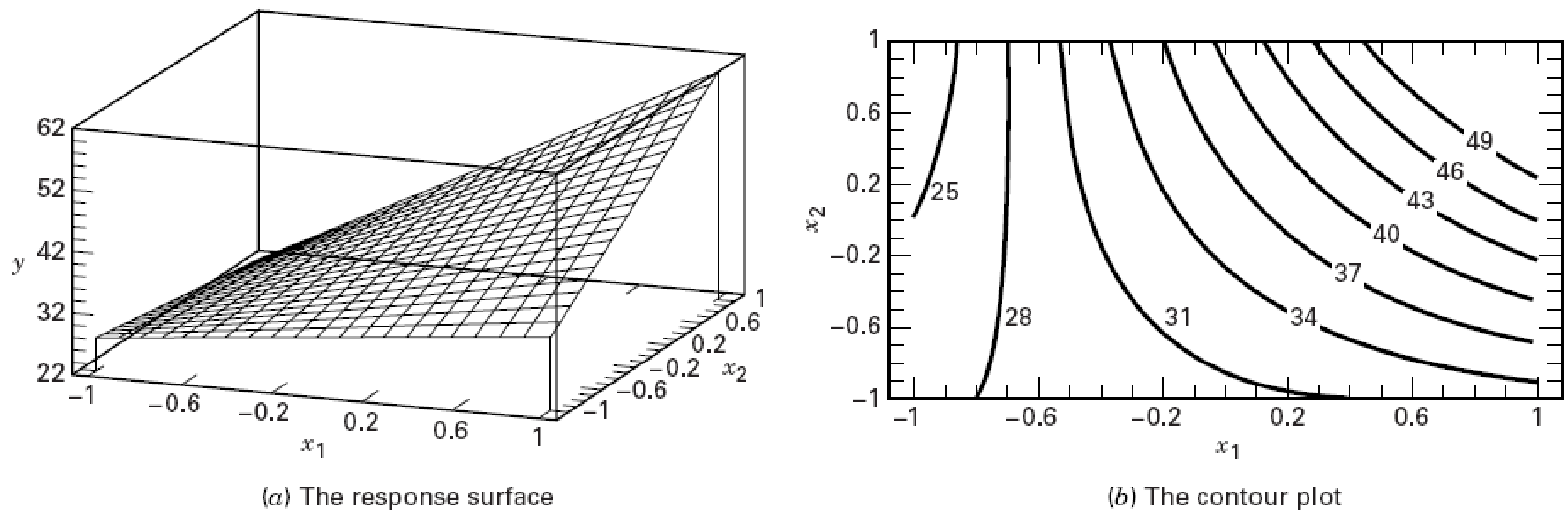
■ **FIGURE 5.5** **Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2$**

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \varepsilon$$

The least squares fit is

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 0.5x_1x_2 \cong 35.5 + 10.5x_1 + 5.5x_2$$

The Effect of Interaction on the Response Surface



■ **FIGURE 5.6** Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 8x_1x_2$

Suppose that we add an interaction term to the model:

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 8x_1x_2$$

Interaction is actually a form of **curvature**

Example 5.1 The Battery Life Experiment Text reference pg. 187

TABLE 5.1
Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F) 15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

A = Material type; *B* = Temperature (A **quantitative** variable)

1. What **effects** do material type & temperature have on life?
2. Is there a choice of material that would give long life ***regardless of temperature*** (a **robust** product)?

The General Two-Factor Factorial Experiment

■ TABLE 5.2

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

a levels of factor *A*; *b* levels of factor *B*; *n* replicates

This is a **completely randomized design**

■ TABLE 5.2

General Arrangement for a Two-Factor Factorial Design

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	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

Statistical (effects) model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

Other models (means model, regression models) can be useful

Extension of the ANOVA to Factorials (Fixed Effects Case) – pg. 189

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2 + an\sum_{j=1}^{b}(\bar{y}_{.j.}-\bar{y}_{...})^2$$

$$+n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2+\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2$$

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

df breakdown:

$$abn-1 = a-1+b-1+(a-1)(b-1)+ab(n-1)$$

ANOVA Table – Fixed Effects Case

■ **TABLE 5.3**

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a - 1$	$MS_A = \frac{SS_A}{a - 1}$	$F_0 = \frac{MS_A}{MS_E}$
B treatments	SS_B	$b - 1$	$MS_B = \frac{SS_B}{b - 1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a - 1)(b - 1)$	$MS_{AB} = \frac{SS_{AB}}{(a - 1)(b - 1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n - 1)$	$MS_E = \frac{SS_E}{ab(n - 1)}$	
Total	SS_T	$abn - 1$		

The sums of squares for the main effects are

$$SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn} \tag{5.7}$$

and

$$SS_B = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{...}^2}{abn} \tag{5.8}$$

It is convenient to obtain the SS_{AB} in two stages. First we compute the sum of squares between the *ab* cell totals, which is called the sum of squares due to "subtotals":

$$SS_{\text{Subtotals}} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{...}^2}{abn}$$

This sum of squares also contains SS_A and SS_B. Therefore, the second step is to compute SS_{AB} as

$$SS_{AB} = SS_{\text{Subtotals}} - SS_A - SS_B \tag{5.9}$$

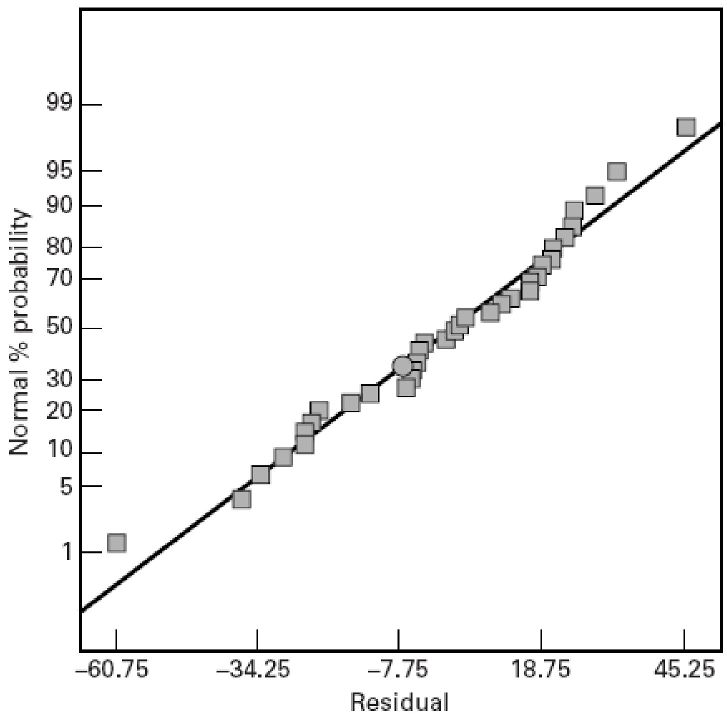
We may compute SS_E by subtraction as

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B \tag{5.10}$$

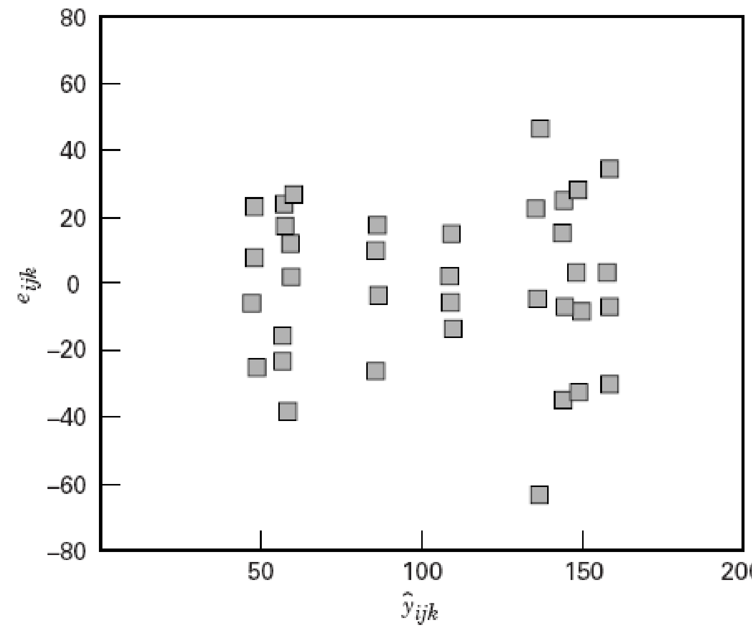
or

$$SS_E = SS_T - SS_{\text{Subtotals}}$$

Residual Analysis – Example 5.1



■ **FIGURE 5.11**
Normal probability plot of residuals for Example 5.1



■ **FIGURE 5.12**
Plot of residuals versus $\hat{y}_{ijk}$ for Example 5.1

Quantitative and Qualitative Factors

- The basic ANOVA procedure treats every factor as if it were **qualitative**
- Sometimes an experiment will involve both **quantitative** and **qualitative** factors, such as in Example 5.1
- This can be accounted for in the analysis to produce **regression models** for the quantitative factors at each level (or combination of levels) of the qualitative factors
- These **response curves** and/or **response surfaces** are often a considerable aid in practical interpretation of the results

EXAMPLE 5.5

The effective life of a cutting tool installed in a numerically controlled machine is thought to be affected by the cutting speed and the tool angle. Three speeds and three angles are selected, and a 3^2 factorial experiment with two replicates is performed. The coded data are shown in Table 5.16. The circled numbers in the cells are the cell totals $\{y_{ij.}\}$.

Table 5.17 shows the JMP out for this experiment. This is a classical ANOVA, treating both factors as categorical. Notice that both design factors tool angle and speed as well as the angle–speed interaction are significant. Since the factors are quantitative, and both factors have three levels, a **second-order model** such as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \varepsilon$$

where x_1 = angle and x_2 = speed could also be fit to the data. The JMP output for this model is shown in Table 5.18.

■ **TABLE 5.16**
Data for Tool Life Experiment

Total Angle (degrees)	**Cutting Speed (in/min)**			
	125	**150**	**175**	$y_{i..}$
15	−2, −1 (−3)	−3, 0 (−3)	2, 3 (5)	−1
20	0, 2 (2)	1, 3 (4)	4, 6 (10)	16
25	−1, 0 (−1)	5, 6 (11)	0, −1 (−1)	9
$y_{.j.}$	−2	12	14	$24 = y_{...}$

Factorials with More Than Two Factors

- Basic procedure is similar to the two-factor case; all *abc…kn* treatment combinations are run in random order
- ANOVA identity is also similar:

$$SS_T = SS_A + SS_B + \cdots + SS_{AB} + SS_{AC} + \cdots$$
$$+ SS_{ABC} + \cdots + SS_{AB\cdots K} + SS_E$$

- Complete three-factor example is in your text book.