however, the variability between looms (σ^2_τ) is not negligible.

12-2 THE TWO-FACTOR FACTORIAL WITH RANDOM FACTORS

Suppose that we have two factors, A and B, and that both factors have a large number of levels that are of interest (as in the previous section, we will assume that the number of levels is infinite). We will choose at random a levels of factor A and b levels of factor B and arrange these factor levels in a factorial experimental design. If the experiment is replicated n times, we may represent the observations by the linear model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \tag{12-15}$$

where the model parameters τ_i, β_j, $(\tau\beta)_{ij}$, and ϵ_{ijk} are all independent random variables. We are also going to assume that the random variables τ_i, β_j, $(\tau\beta)_{ij}$, and ϵ_{ijk} are normally distributed with mean zero and variances given by $V(\tau_i) = \sigma_\tau^2$, $V(\beta_j) = \sigma_\beta^2$, $V[(\tau\beta)_{ij}] = \sigma_{\tau\beta}^2$, and $V(\epsilon_{ijk}) = \sigma^2$. Therefore the variance of any observation is

$$V(y_{ijk}) = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2 \tag{12-16}$$

and σ_τ^2, σ_β^2, $\sigma_{\tau\beta}^2$, and σ^2 are the **variance components**. The hypotheses that we are interested in testing are $H_0\colon\sigma_\tau^2 = 0$, $H_0\colon\sigma_\beta^2 = 0$, and $H_0\colon\sigma_{\tau\beta}^2 = 0$. Notice the similarity to the single-factor random effects model.

The numerical calculations in the analysis of variance remains unchanged; that is, SS_A, SS_B, SS_{AB}, SS_T, and SS_E are all calculated as in the fixed effects case. However, to form the test statistics, we must examine the **expected mean squares**. It may be shown that

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma_{\tau\beta}^2 + bn\sigma_\tau^2 \\ E(MS_B) &= \sigma^2 + n\sigma_{\tau\beta}^2 + an\sigma_\beta^2 \\ E(MS_{AB}) &= \sigma^2 + n\sigma_{\tau\beta}^2 \end{aligned} \tag{12-17}$$

and

$$E(MS_E) = \sigma^2$$

From the expected mean squares, we see that the appropriate statistic for testing the no-interaction hypothesis $H_0\colon\sigma_{\tau\beta}^2 = 0$ is

$$F_0 = \frac{MS_{AB}}{MS_E} \tag{12-18}$$

$$F_0 = \frac{MS_{AB}}{MS_E} \tag{12-18}$$

because under H_0 both numerator and denominator of F_0 have expectation σ^2, and only if H_0 is false is $E(MS_{AB})$ greater than $E(MS_E)$. The ratio F_0 is distributed as $F_{(a-1)(b-1),ab(n-1)}$. Similarly, for testing $H_0: \sigma_\tau^2 = 0$ we would use

$$F_0 = \frac{MS_A}{MS_{AB}} \tag{12-19}$$

which is distributed as $F_{a-1,(a-1)(b-1)}$, and for testing $H_0: \sigma_\beta^2 = 0$ the statistic is

$$F_0 = \frac{MS_B}{MS_{AB}} \tag{12-20}$$

which is distributed as $F_{b-1,(a-1)(b-1)}$. These are all upper-tail, one-tail tests. Notice that these test statistics are not the same as those used if both factors A and B are fixed. The expected mean squares are always used as a guide to test statistic construction.

In many experiments involving random factors, interest centers at least as much on estimating the variance components as on hypothesis testing. The variance components may be estimated by the **analysis of variance method**, that is, by equating the observed mean squares in the lines of the analysis of variance table to their expected values and solving for the variance components. This yields

$$\begin{aligned}
\hat{\sigma}^2 &= MS_E \\
\hat{\sigma}^2_{\tau\beta} &= \frac{MS_{AB} - MS_E}{n} \\
\hat{\sigma}^2_{\beta} &= \frac{MS_B - MS_{AB}}{an} \\
\hat{\sigma}^2_{\tau} &= \frac{MS_A - MS_{AB}}{bn}
\end{aligned} \qquad (12\text{-}21)$$

as the point estimates of the variance components in the two-factor random effects model. We will discuss other methods for obtaining point estimates of the variance components and procedures for constructing confidence intervals in Section 12-7.

EXAMPLE 12-2 ....

A Measurement Systems Capability Study

Statistically designed experiments are frequently used to investigate the sources of variability that affect a system. A common industrial application is to use a designed experiment to study the components of variability in a measurement system. These studies are often called **gauge capability studies** or **gauge repeatability and reproducibility (R&R) studies**, because these are the components of variability that are of interest.

A typical gauge R&R experiment [from Montgomery (1996)] is shown in Table 12-3. An instrument or gauge is used to measure a critical dimension on a part. Twenty parts have been selected from the production process, and three randomly selected operators measure each part twice with this gauge. The order in which the measurements are made is completely randomized, so this is a two-factor factorial experiment with design factors parts and operators, with two replications. Both parts and operators are random factors. The variance component identity in Equation 12-15 applies; namely,

$$\sigma_y^2 = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$$

where σ_y^2 is the total variability (including variability due to the different parts, variability due to the different operators, and variability due to the gauge), σ_τ^2 is the variance component for parts, σ_β^2 is the variance component for operators, $\sigma_{\tau\beta}^2$ is the variance component that represents interaction between parts and operators, and σ^2 is the random

Table 12-3 The Measurement Systems Capability Experiment in Example 12-2

Part Number	Operator 1		Operator 2		Operator 3	
1	21	20	20	20	19	21
2	24	23	24	24	23	24
3	20	21	19	21	20	22
4	27	27	28	26	27	28
5	19	18	19	18	18	21
6	23	21	24	21	23	22
7	22	21	22	24	22	20
8	19	17	18	20	19	18
9	24	23	25	23	24	24
10	25	23	26	25	24	25
11	21	20	20	20	21	20
12	18	19	17	19	18	19
13	23	25	25	25	25	25
14	24	24	23	25	24	25
15	29	30	30	28	31	30
16	26	26	25	26	25	27
17	20	20	19	20	20	20
18	19	21	19	19	21	23
19	25	26	25	24	25	25
20	19	19	18	17	19	17

experimental error. Typically, the variance component σ^2 is called the gauge repeatability, because σ^2 can be thought of as reflecting the variation observed when the same part is measured by the same operator, and

$$\sigma_\beta^2 + \sigma_{\tau\beta}^2$$

is usually called the reproducibility of the gauge, because it reflects the additional variability in the measurement system resulting from use of the instrument by the operator. These experiments are usually performed with the objective of estimating the variance components.

Table 12-4 (on the facing page) shows the analysis of variance for this experiment. The computations were performed using the Balanced ANOVA routine in Minitab. Based on the P-values, we conclude that the effect of parts is large, operators may have a small effect, and that there is no significant part–operator interaction. We may use Equation 12-21 to estimate the variance components as follows:

$$\hat{\sigma}_\tau^2 = \frac{62.39 - 0.71}{(3)(2)} = 10.28$$

$$\hat{\sigma}_\beta^2 = \frac{1.31 - 0.71}{(20)(2)} = 0.015$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{0.71 - 0.99}{2} = -0.14$$

and

$$\hat{\sigma}^2 = 0.99$$

$$\hat{\sigma}^2 = 0.99$$

The bottom portion of the Minitab output in Table 12-4 contains the expected mean squares for the random model with numbers in parentheses representing the variance components [(4) represents σ^2, (3) represents $\sigma^2_{\tau\beta}$, etc.]. The estimates of the variance components are also given, along with the error term that was used in testing that variance component in the analysis of variance. We will discuss the terminology **unrestricted model** later; it has no relevance in random models.

Notice that the estimate of one of the variance components, $\hat{\sigma}^2_{\tau\beta}$, is negative. This is certainly not reasonable because by definition variances are nonnegative. Unfortunately, negative estimates of variance components can result when we use the analysis of variance method of estimation (this is considered one of its drawbacks). There are a variety of ways to deal with this. One possibility is to assume that the negative estimate means that the variance component is really zero and just set it to zero, leaving the other nonnegative estimates unchanged. Another approach is to estimate the variance components with a method that assures nonnegative estimates (we will discuss this briefly in Section 12-7). Finally, we could note that the P-value for the interaction term in Table 12-4 is very large, take this as evidence that $\sigma^2_{\tau\beta}$ really is zero, that there is no interaction effect, and fit a **reduced model** of the form

$$y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$$

that does not include the interaction term. This is a relatively easy approach and one that often works nearly as well as more sophisticated methods.

Table 12-4 Analysis of Variance (Minitab Balanced ANOVA) for Example 12-2

Analysis of Variance (Balanced Designs)

```
Factor      Type Levels Values
part      random     20      1      2      3      4      5      6      7
                             8      9     10     11     12     13     14
                            15     16     17     18     19     20
operator random       3      1      2      3

Analysis of Variance for y

Source            DF        SS        MS       F       P
part              19  1185.425    62.391   87.65   0.000
operator           2     2.617     1.308    1.84   0.173
part*operator     38    27.050     0.712    0.72   0.861
Error             60    59.500     0.992
Total            119  1274.592

Source               Variance  Error  Expected Mean Square for Each Term
                    component  term   (using unrestricted model)
 1 part               10.2798    3     (4) + 2(3) + 6(1)
 2 operator            0.0149    3     (4) + 2(3) + 40(2)
 3 part*operator      -0.1399    4     (4) + 2(3)
 4 Error               0.9917          (4)
```

Table 12-5 Analysis of Variance for the Reduced Model, Example 12-2

Analysis of Variance (Balanced Designs)

```
Factor      Type  Levels Values
part      random     20      1      2      3      4      5      6      7
                             8      9     10     11     12     13     14
                            15     16     17     18     19     20
operator random       3      1      2      3

Analysis of Variance for y

Source        DF          SS          MS         F        P
part          19    1185.425      62.391     70.64    0.000
operator       2       2.617       1.308      1.48    0.232
Error         98      86.550       0.883
Total        119    1274.592

Source            Variance   Error   Expected Mean Square for Each Term
                  component  term     (using unrestricted model)
  1 part           10.2513     3      (3) + 6(1)
  2 operator        0.0106     3      (3) + 40(2)
  3 Error           0.8832            (3)
```

there is no interaction term in the model, both main effects are tested against the error term, and the estimates of the variance components are

$$\hat{\sigma}_\tau^2 = \frac{62.39 - 0.88}{(3)(2)} = 10.25$$

$$\hat{\sigma}_\beta^2 = \frac{1.31 - 0.88}{(20)(2)} = 0.0108$$

$$\hat{\sigma}^2 = 0.88$$

Finally, we could estimate the variance of the gauge as the sum of the variance component estimates $\hat{\sigma}^2$ and $\hat{\sigma}^2_\beta$ as

$$\begin{aligned}\hat{\sigma}^2_{\text{gauge}} &= \hat{\sigma}^2 + \hat{\sigma}^2_\beta \\ &= 0.88 + 0.0108 \\ &= 0.8908\end{aligned}$$

The variability in the gauge appears small relative to the variability in the product. This is generally a desirable situation, implying that the gauge is capable of distinguishing among different grades of product.

12-3 THE TWO-FACTOR MIXED MODEL

We now consider the situation where one of the factors A is fixed and the other B is random. This is called the **mixed model** analysis of variance. The linear statistical model is

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \tag{12-22}$$

Here τ_i is a fixed effect, β_j is a random effect, the interaction $(\tau\beta)_{ij}$ is assumed to be a random effect, and ϵ_{ijk} is a random error. We also assume that the $\{\tau_i\}$ are fixed effects such that $\sum_{i=1}^{a}\tau_i = 0$ and β_j is a NID$(0, \sigma_\beta^2)$ random variable. The interaction effect, $(\tau\beta)_{ij}$, is a normal random variable with mean 0 and variance $[(a-1)/a]\sigma_{\tau\beta}^2$; however, summing the interaction component over the fixed factor equals zero. That is,

$$\sum_{i=1}^{a} (\tau\beta)_{ij} = (\tau\beta)_{.j} = 0 \qquad j = 1, 2, \ldots, b$$

This restriction implies that certain interaction elements at different levels of the fixed factor are not independent. In fact, we may show (see Problem 12-25) that

$$\text{Cov}[(\tau\beta)_{ij}, (\tau\beta)_{i'j}] = -\frac{1}{a}\,\sigma_{\tau\beta}^2 \qquad i \neq i'$$

The covariance between $(\tau\beta)_{ij}$ and $(\tau\beta)_{ij'}$ for $j \neq j'$ is zero, and the random error ϵ_{ijk} is NID$(0, \sigma^2)$. Because the sum of the interaction effects over the levels of the fixed factor equals zero, this version of the mixed model is often called the **restricted model**.

equals zero, this version of the mixed model is often called the **restricted model**.

In this model the variance of $(\tau\beta)_{ij}$ is defined as $[(a-1)/a]\sigma^2_{\tau\beta}$ rather than $\sigma^2_{\tau\beta}$ to simplify the expected mean squares. The assumption $(\tau\beta)_{.j} = 0$ also has an effect on the expected mean squares, which we may show are

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn \sum_{i=1}^{a} \tau_i^2}{a-1} \\ E(MS_B) &= \sigma^2 + an\sigma^2_{\beta} \\ E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \end{aligned} \tag{12-23}$$

and

$$E(MS_E) = \sigma^2$$

Therefore, the appropriate test statistic for testing that the means of the fixed factor effects are equal, or $H_0: \tau_i = 0$, is

$$F_0 = \frac{MS_A}{MS_{AB}}$$

for which the reference distribution is $F_{a-1,(a-1)(b-1)}$. For testing $H_0: \sigma^2_{\beta} = 0$, the test statistic is

$$F_0 = \frac{MS_B}{MS_E}$$

with reference distribution $F_{b-1,ab(n-1)}$. Finally, for testing the interaction hypothesis $H_0: \sigma^2_{\tau\beta} = 0$, we would use

$$F_0 = \frac{MS_{AB}}{MS_E}$$

which has reference distribution $F_{(a-1)(b-1),ab(n-1)}$.

In the mixed model, it is possible to estimate the fixed factor effects as

$$\begin{aligned} \hat{\mu} &= \bar{y}_{...} \\ \hat{\tau}_i &= \bar{y}_{i..} - \bar{y}_{...} \qquad i = 1, 2, \ldots, a \end{aligned} \tag{12-24}$$

The variance components σ_β^2, $\sigma_{\tau\beta}^2$, and σ^2 may be estimated using the analysis of variance method. Eliminating the first equation from Equations 12-23 leaves three equations in three unknowns, whose solutions are

$$\hat{\sigma}_\beta^2 = \frac{MS_B - MS_E}{an}$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n} \tag{12-25}$$

and

$$\hat{\sigma}^2 = MS_E$$

This general approach can be used to estimate the variance components in *any* mixed model. After eliminating the mean squares containing fixed factors, there will always be a set of equations remaining that can be solved for the variance components.

In mixed models the experimenter may be interested in testing hypotheses or constructing confidence intervals about individual treatment means for the fixed factor. In using such procedures, care must be exercised to use the proper standard error of the treatment mean. The standard error of the fixed effect treatment mean is

$$\left[\frac{\text{Mean square for testing the fixed effect}}{\text{Number of observations in each treatment mean}}\right]^{1/2} = \sqrt{\frac{MS_{AB}}{bn}}$$

Notice that this is just the standard error that we would use if this was a fixed effects model, except that MS_E has been replaced by the mean square used for hypothesis testing.

EXAMPLE 12-3

The Measurement Systems Capability Experiment Revisited

Reconsider the gauge R&R experiment described in Example 12-2. Suppose now that there are only three operators that use this gauge, so the operators are a fixed factor. However, because the parts are chosen at random, the experiment now involves a mixed model.

The analysis of variance for the mixed model is shown in Table 12-6 on the facing page. The computations were performed using the Balanced ANOVA routine in Minitab. We specified that the restricted model be used in the Minitab analysis. Minitab also generated the expected mean squares for this model. In the Minitab output, the quantity $Q[2]$ indicates a quadratic expression involving the fixed factor effect operator. That is, $Q[2] = \Sigma_{j=1}^{b} \beta_j^2/(b - 1)$. The conclusions are similar to Example 12-2. The variance components may be estimated from Equation (12-25) as

$$\hat{\sigma}^2_{\text{Parts}} = \frac{MS_{\text{Parts}} - MS_E}{an} = \frac{62.39 - 0.99}{(3)(2)} = 10.23$$

$$\hat{\sigma}^2_{\text{Parts}\times\text{operators}} = \frac{MS_{\text{Parts}\times\text{operators}} - MS_E}{n} = \frac{0.71 - 0.99}{2} = -0.14$$

$$\hat{\sigma}^2 = MS_E = 0.99$$

These results are also given in the Minitab output. Once again, a negative estimate of the interaction variance component results. An appropriate course of action would be to

Table 12-6 Analysis of Variance (Minitab) for the Mixed Model in Example 12-3. The Restricted Model is Assumed.

Analysis of Variance (Balanced Designs)

```
Factor      Type Levels Values
part      random     20      1     2     3     4     5     6     7
                             8     9    10    11    12    13    14
                            15    16    17    18    19    20
operator fixed        3      1     2     3

Analysis of Variance for y

Source            DF        SS        MS       F       P
part              19  1185.425    62.391   62.92   0.000
operator           2     2.617     1.308    1.84   0.173
part*operator     38    27.050     0.712    0.72   0.861
Error             60    59.500     0.992
Total            119  1274.592

Source              Variance  Error  Expected Mean Square for Each Term
                   component  term   (using restricted model)
 1 part              10.2332    4     (4) + 6(1)
 2 operator                     3     (4) + 2(3) + 40Q[2]
 3 part*operator     -0.1399    4     (4) + 2(3)
 4 Error              0.9917          (4)
```

12-4 SAMPLE SIZE DETERMINATION WITH RANDOM EFFECTS

The operating characteristic curves in the Appendix may be used for sample size determination in experiments with random factors. We begin with the single-factor random effects model of Section 12-1. The type II error probability for the random effects model is

$$\begin{aligned}\beta &= 1 - P\{\text{Reject } H_0 | H_0 \text{ is false}\} \\ &= 1 - P\{F_0 > F_{\alpha,a-1,N-a} | \sigma_\tau^2 > 0\}\end{aligned} \tag{12-28}$$

Once again, the distribution of the test statistic $F_0 = MS_{\text{Treatments}}/MS_E$ under the alternative hypothesis is needed. It can be shown that if H_1 is true ($\sigma_\tau^2 > 0$), the distribution of F_0 is central F with $a - 1$ and $N - a$ degrees of freedom.

Because the type II error probability of the random effects model is based on the usual central F distribution, we could use the tables of the F distribution in the Appendix to evaluate Equation 12-28. However, it is simpler to determine the sensitivity of the test through the use of operating characteristic curves. A set of these curves for various values of numerator degrees of freedom, denominator degrees of freedom, and α of 0.05 or 0.01 is provided in Chart VI of the Appendix. These curves plot the probability of type II error against the parameter λ, where

$$\lambda = \sqrt{1 + \frac{n\sigma_\tau^2}{\sigma^2}} \tag{12-29}$$

Note that λ involves two unknown parameters, σ^2 and σ_τ^2. We may be able to estimate σ_τ^2 if we have an idea about how much variability in the population of treatments it is important to detect. An estimate of σ^2 may be chosen using prior experience or judgment. Sometimes it is helpful to define the value of σ_τ^2 we are interested in detecting in terms of the ratio σ_τ^2/σ^2.

EXAMPLE 12-5 ··

Suppose we have five treatments selected at random with six observations per treatment and $\alpha = 0.05$, and we wish to determine the power of the test if σ_τ^2 is equal to σ^2. Because $a = 5$, $n = 6$, and $\sigma_\tau^2 = \sigma^2$, we may compute

$$\lambda = \sqrt{1 + 6(1)} = 2.646$$

From the operating characteristic curve with $a - 1 = 4$, $N - a = 25$ degrees of freedom, and $\alpha = 0.05$, we find that

$$\beta \simeq 0.20$$

and thus the power is approximately 0.80.

··

We can also use the percentage increase in the standard deviation of an observation method to determine sample size. If the treatments are homogeneous, then the standard deviation of an observation selected at random is σ. However, if the treatments are different, the standard deviation of a randomly chosen observation is

$$\sqrt{\sigma^2 + \sigma_\tau^2}$$

If P is the fixed percentage increase in the standard deviation of an observation beyond which rejection of the null hypothesis is desired,

$$\frac{\sqrt{\sigma^2 + \sigma_\tau^2}}{\sigma} = 1 + 0.01P$$

or

$$\frac{\sigma_\tau^2}{\sigma^2} = (1 + 0.01P)^2 - 1$$

Therefore, using Equation 12-29, we find that

$$\lambda = \sqrt{1 + \frac{n\sigma_\tau^2}{\sigma^2}} = \sqrt{1 + n[(1 + 0.01P)^2 - 1]} \tag{12-30}$$

For a given P, the operating characteristic curves in Appendix Chart VI can be used to find the desired sample size.

We can also use the operating characteristic curves for sample size determination for the two-factor random effects model and the mixed model. Appendix Chart VI is used for the random effects model. The parameter λ, numerator degrees of freedom, and denominator degrees of freedom are shown in the top half of Table 12-8. For the mixed model, both Charts V and VI in the Appendix must be used. The appropriate values for Φ^2 and λ are shown in the bottom half of Table 12-8.

Table 12-8 Operating Characteristic Curve Parameters for Tables V and VI of the Appendix for the Two-Factor Random Effects and Mixed Models

The Random Effects Model

Factor	λ	Numerator Degrees of Freedom	Denominator Degrees of Freedom
A	$\sqrt{1 + \dfrac{bn\sigma_\tau^2}{\sigma^2 + n\sigma_{\tau\beta}^2}}$	$a - 1$	$(a - 1)(b - 1)$
B	$\sqrt{1 + \dfrac{an\sigma_\beta^2}{\sigma^2 + n\sigma_{\tau\beta}^2}}$	$b - 1$	$(a - 1)(b - 1)$
AB	$\sqrt{1 + \dfrac{n\sigma_{\tau\beta}^2}{\sigma^2}}$	$(a - 1)(b - 1)$	$ab(n - 1)$

The Mixed Model

Factor	Parameter	Numerator Degrees of Freedom	Denominator Degrees of Freedom	Appendix Chart
A (Fixed)	$\Phi^2 = \dfrac{bn \sum_{i=1}^{a} \tau_i^2}{a[\sigma^2 + n\sigma_{\tau\beta}^2]}$	$a - 1$	$(a - 1)(b - 1)$	V
B (Random)	$\lambda = \sqrt{1 + \dfrac{an\sigma_\beta^2}{\sigma^2}}$	$b - 1$	$ab(n - 1)$	VI
AB	$\lambda = \sqrt{1 + \dfrac{n\sigma_{\tau\beta}^2}{\sigma^2}}$	$(a - 1)(b - 1)$	$ab(n - 1)$	VI