Design of Engineering Experiments – The Blocking Principle

- Text Reference, Chapter 4
- **Blocking** and **nuisance factors**
- The randomized complete block design or the **RCBD**
- Extension of the ANOVA to the RCBD
- Other blocking scenarios…Latin square designs

The Blocking Principle

- **Blocking** is a technique for dealing with **nuisance factors**
- A **nuisance** factor is a factor that probably has some effect on the response, but it's of no interest to the experimenter…however, the variability it transmits to the response needs to be minimized
- Typical nuisance factors include batches of raw material, operators, pieces of test equipment, time (shifts, days, etc.), different experimental units
- **Many** industrial experiments involve blocking (or should)
- Failure to block is a common flaw in designing an experiment (consequences?)

The Blocking Principle

- If the nuisance variable is **known** and **controllable**, we use **blocking**
- If the nuisance factor is **known** and **uncontrollable**, sometimes we can use the **analysis of covariance** (see Chapter 15) to remove the effect of the nuisance factor from the analysis
- If the nuisance factor is **unknown** and **uncontrollable** (a **"lurking" variable**), we hope that **randomization** balances out its impact across the experiment
- Sometimes several sources of variability are **combined** in a block, so the block becomes an aggregate variable

The Hardness Testing Example

- Text reference, pg 139, 140
- We wish to determine whether 4 different tips produce different (mean) hardness reading on a Rockwell hardness tester
- Gauge & measurement systems capability studies are frequent areas for applying DOX
- Assignment of the tips to an **experimental unit**; that is, a test coupon
- Structure of a completely randomized experiment
- The test coupons are a source of **nuisance variability**
- Alternatively, the experimenter may want to test the tips across coupons of various hardness levels
- The need for blocking

The Hardness Testing Example

- To conduct this experiment as a RCBD, assign all 4 tips to each coupon
- Each coupon is called a "**block**"; that is, it's a more homogenous experimental unit on which to test the tips
- Variability **between** blocks can be large, variability **within** a block should be relatively small
- In general, a **block** is a specific level of the nuisance factor
- A complete replicate of the basic experiment is conducted in each block
- A block represents a **restriction on randomization**
- All runs **within** a block are **randomized**

To illustrate the general idea, suppose we wish to determine whether or not four different tips produce different readings on a hardness testing machine. An experiment such as this might be part of a gauge capability study. The machine operates by pressing the tip into a metal test coupon, and from the depth of the resulting depression, the hardness of the coupon can be determined. The experimenter has decided to obtain four observations for each tip. There is only one factor—tip type—and a completely randomized single-factor design would consist of randomly assigning each one of the $4 \times 4 = 16$ runs to an **experimental unit**, that is, a metal coupon, and observing the hardness reading that results. Thus, 16 different metal test coupons would be required in this experiment, one for each run in the design.

There is a potentially serious problem with a completely randomized experiment in this design situation. If the metal coupons differ slightly in their hardness, as might

Table 4-1 Randomized Complete Block Design for the Hardness Testing Experiment

	Test Coupon			
Type of Tip	1	2	3	4
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

happen if they are taken from ingots that are produced in different heats, the experimental units (the coupons) will contribute to the variability observed in the hardness data. As a result, the experimental error will reflect *both* random error *and* variability between coupons.

We would like to make the experimental error as small as possible; that is, we would like to remove the variability between coupons from the experimental error. A design that would accomplish this requires the experimenter to test each tip once on each of four coupons. This design, shown in Table 4-1, is called a **randomized complete block design (RCBD)**. The observed response is the Rockwell C scale hardness minus 40.

The Hardness Testing Example

- Suppose that we use $b = 4$ blocks:

■ TABLE 4.1
Randomized Complete Block Design for the Hardness Testing Experiment

Test Coupon (Block)			
1	2	3	4
Tip 3	Tip 3	Tip 2	Tip 1
Tip 1	Tip 4	Tip 1	Tip 4
Tip 4	Tip 2	Tip 3	Tip 2
Tip 2	Tip 1	Tip 4	Tip 3

- Notice the **two-way structure** of the experiment
- Once again, we are interested in testing the equality of treatment means, but now we have to remove the variability associated with the nuisance factor (the blocks)

Extension of the ANOVA to the RCBD

- Suppose that there are *a* treatments (factor levels) and *b* blocks
- A **statistical model** (effects model) for the RCBD is

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

- The relevant (fixed effects) hypotheses are

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \text{ where } \mu_i = (1/b)\sum_{j=1}^{b} (\mu + \tau_i + \beta_j) = \mu + \tau_i$$

Extension of the ANOVA to the RCBD

ANOVA partitioning of total variability:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}[(\bar{y}_{i.}-\bar{y}_{..})+(\bar{y}_{.j}-\bar{y}_{..})$$

$$+(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})]^2$$

$$= b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$$

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

Extension of the ANOVA to the RCBD

The degrees of freedom for the sums of squares in

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

are as follows:

$$ab-1 = a-1+b-1+(a-1)(b-1)$$

Therefore, ratios of sums of squares to their degrees of freedom result in mean squares and the ratio of the mean square for treatments to the error mean square is an *F* statistic that can be used to test the hypothesis of equal treatment means

ANOVA Display for the RCBD

■ **TABLE 4.2**

Analysis of Variance for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$\frac{SS_{\text{Treatments}}}{a - 1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b - 1$	$\frac{SS_{\text{Blocks}}}{b - 1}$	
Error	SS_E	$(a - 1)(b - 1)$	$\frac{SS_E}{(a - 1)(b - 1)}$	
Total	SS_T	$N - 1$		

Manual computing:

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N} \quad \textbf{(4.9)}$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N} \quad \textbf{(4.10)}$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N} \quad \textbf{(4.11)}$$

and the error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}} \quad \textbf{(4.12)}$$

EXAMPLE 4-1 ....

Consider the hardness testing experiment described in Section 4-1. There are four tips and four available metal coupons. Each tip is tested once on each coupon, resulting in a randomized complete block design. The data obtained are repeated for convenience in Table 4-3. Remember that the *order* in which the tips were tested on a particular coupon was determined randomly. To simplify the calculations, we code the original data by

Table 4-3 Randomized Complete Block Design for the Hardness Testing Experiment

	Coupon (Block)			
Type of Tip	1	2	3	4
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

Table 4-4 Coded Data for the Hardness Testing Experiment

Type of Tip	Coupon (Block) 1	2	3	4	$y_{i.}$
1	−2	−1	1	5	3
2	−1	−2	3	4	4
3	−3	−1	0	2	−2
4	2	1	5	7	15
$y_{.j}$	−4	−3	9	18	$20 = y_{..}$

subtracting 9.5 from each observation and multiplying the result by 10. This yields the data in Table 4-4. The sums of squares are obtained as follows:

$$SS_T = \sum_{i=1}^{4}\sum_{j=1}^{4} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= 154.00 - \frac{(20)^2}{16} = 129.00$$

$$SS_{\text{Treatments}} = \frac{1}{b}\sum_{i=1}^{4} y_{i.}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{4}[(3)^2 + (4)^2 + (-2)^2 + (15)^2] - \frac{(20)^2}{16} = 38.50$$

$$SS_{\text{Blocks}} = \frac{1}{a}\sum_{j=1}^{4} y_{.j}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{4}[(-4)^2 + (-3)^2 + (9)^2 + (18)^2] - \frac{(20)^2}{16} = 82.50$$

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$

$$= 129.00 - 38.50 - 82.50 = 8.00$$

The analysis of variance is shown in Table 4-5. Using $\alpha = 0.05$, the critical value of F is $F_{0.05,3,9} = 3.86$. Because $14.44 > 3.86$, we conclude that the type of tip affects the mean hardness reading. The P-value for the test is also quite small. Also, the coupons (blocks) seem to differ significantly, because the mean square for blocks is large relative to error.

It is interesting to observe the results we would have obtained had we not been aware of randomized block designs. Suppose we used four coupons, randomly assigned the tips to each, and (by chance) the same design resulted as in Table 4-3. The incorrect analysis of these data as a completely randomized single-factor design is shown in Table 4-6.

Table 4-5 Analysis of Variance for the Hardness Testing Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Treatments (type of tip)	38.50	3	12.83	14.44	0.0009
Blocks (coupons)	82.50	3	27.50		
Error	8.00	9	0.89		
Total	129.00	15			

Table 4-6 Incorrect Analysis of the Hardness Testing Experiment as a Completely Randomized Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Type of tip	38.50	3	12.83	1.70
Error	90.50	12	7.54	
Total	129.00	15		

Because $F_{0.05,3,12} = 3.49$, the hypothesis of equal mean hardness measurements from the four tips cannot be rejected. Thus, the randomized block design reduces the amount of noise in the data sufficiently for differences among the four tips to be detected. This illustrates a very important point. If an experimenter fails to block when he or she should have, the effect may be to inflate the experimental error so much that important differences among the treatment means may be undetectable.

Other Aspects of the RCBD
See Text, Section 4.1.3, pg. 132

- The RCBD utilizes an **additive model** – no interaction between treatments and blocks
- Treatments and/or blocks as random effects
- Missing values
- What are the **consequences** of **not blocking** if we should have?
- **Sample sizing** in the RCBD? The **OC curve** approach can be used to determine the number of blocks to run.

Random Blocks and/or Treatments

Assuming that the RCBD model Equation 4.1 is appropriate, if the blocks are random and the treatments are fixed we can show that:

$$\begin{aligned} E(y_{ij}) &= \mu + \tau_i, \qquad i = 1, 2, \ldots, a \\ V(y_{ij}) &= \sigma_\beta^2 + \sigma^2 \\ Cov(y_{ij}, y_{i'j'}) &= 0, \quad j \neq j' \\ Cov(y_{ij}, y_{i'j}) &= \sigma_\beta^2 \quad i \neq i' \end{aligned} \tag{4.14}$$

Thus, the variance of the observations is constant, the covariance between any two observations in different blocks is zero, but the covariance between two observations from the same block is σ_β^2. The expected mean squares from the usual ANOVA partitioning of the total sum of squares are

$$\begin{aligned} E(MS_{\text{Treatments}}) &= \sigma^2 + \frac{b\sum_{i=1}^{a}\tau_i^2}{a-1} \\ E(MS_{\text{Blocks}}) &= \sigma^2 + a\sigma_\beta^2 \\ E(MS_E) &= \sigma^2 \end{aligned} \tag{4.15}$$

The appropriate statistic for testing the null hypothesis of no treatment effects (all $\tau_i = 0$) is

$$F_0 = \frac{MS_{\text{Treatment}}}{MS_E}$$

which is exactly the same test statistic we used in the case where the blocks were fixed. Based on the expected mean squares, we can obtain an ANOVA-type estimator of the variance component for blocks as

$$\hat{\sigma}_\beta^2 = \frac{MS_{\text{Blocks}} - MS_E}{a} \tag{4.16}$$

For example, for the vascular graft experiment in Example 4.1 the estimate of σ_β^2 is

$$\hat{\sigma}_\beta^2 = \frac{MS_{\text{Blocks}} - MS_E}{a} = \frac{38.45 - 7.33}{4} = 7.78$$

4-1.2 Model Adequacy Checking

We have previously discussed the importance of checking the adequacy of the assumed model. Generally, we should be alert for potential problems with the normality assumption, unequal error variance by treatment or block, and block–treatment interaction. As in the completely randomized design, residual analysis is the major tool used in this diagnostic checking. The residuals for the randomized block design in Example 4-1 are listed at the bottom of the Design-Expert output in Figure 4-2. The coded residuals would be found by multiplying these residuals by 10. The observations, fitted values, and residuals for the coded hardness testing data in Example 4-1 are as follows:

y_{ij}	$\hat{y}_{ij}$	e_{ij}
−2.00	−1.50	−0.50
−1.00	−1.25	0.25
1.00	1.75	−0.75
5.00	4.00	1.00
−1.00	−1.25	0.25
−2.00	−1.00	−1.00
3.00	2.00	1.00
4.00	4.25	−0.25
−3.00	−2.75	−0.25
−1.00	−2.50	1.50
0.00	0.50	−0.50
2.00	2.75	−0.75
2.00	1.50	0.50
1.00	1.75	−0.75
5.00	4.75	0.25
7.00	7.00	0.00

A normal probability plot and a dot diagram of these residuals are shown in Figure 4-4 on page 136. There is no severe indication of nonnormality, nor is there any evidence pointing to possible outliers. Figure 4-5 (page 137) shows plots of the residuals by type

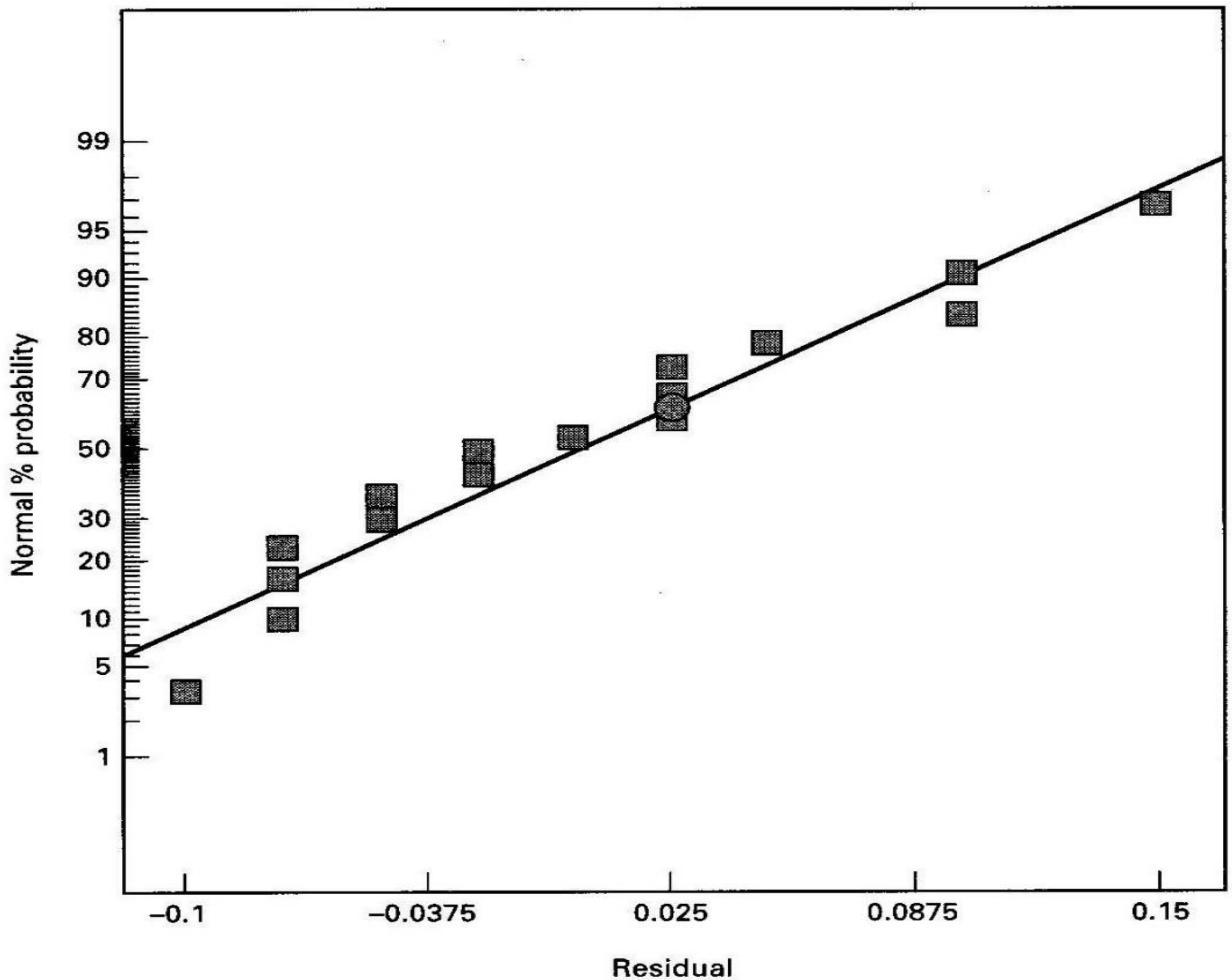

Figure 4-4 Normal probability plot of residuals for Example 4-1.

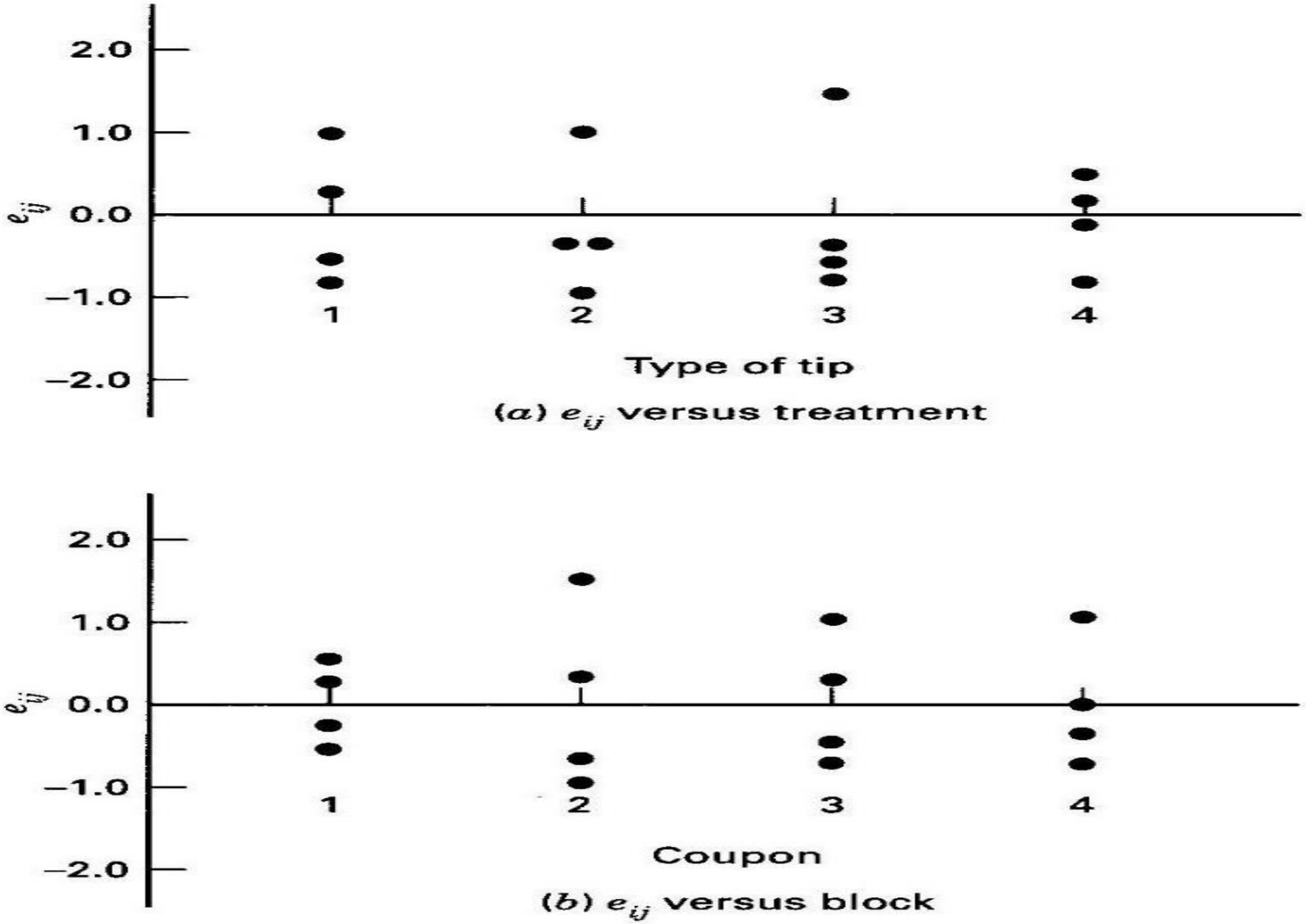

Figure 4-5 Plot of residuals by tip type (treatment) and by coupon (block) for Example 4-1.

is completely **additive**. This says that, for example, if the first treatment causes the expected response to increase by five units ($\tau_1 = 5$) and if the first block increases the expected response by 2 units ($\beta_1 = 2$), the expected increase in response of *both* treatment 1 *and* block 1 together is $E(y_{11}) = \mu + \tau_1 + \beta_1 = \mu + 5 + 2 = \mu + 7$. In general,

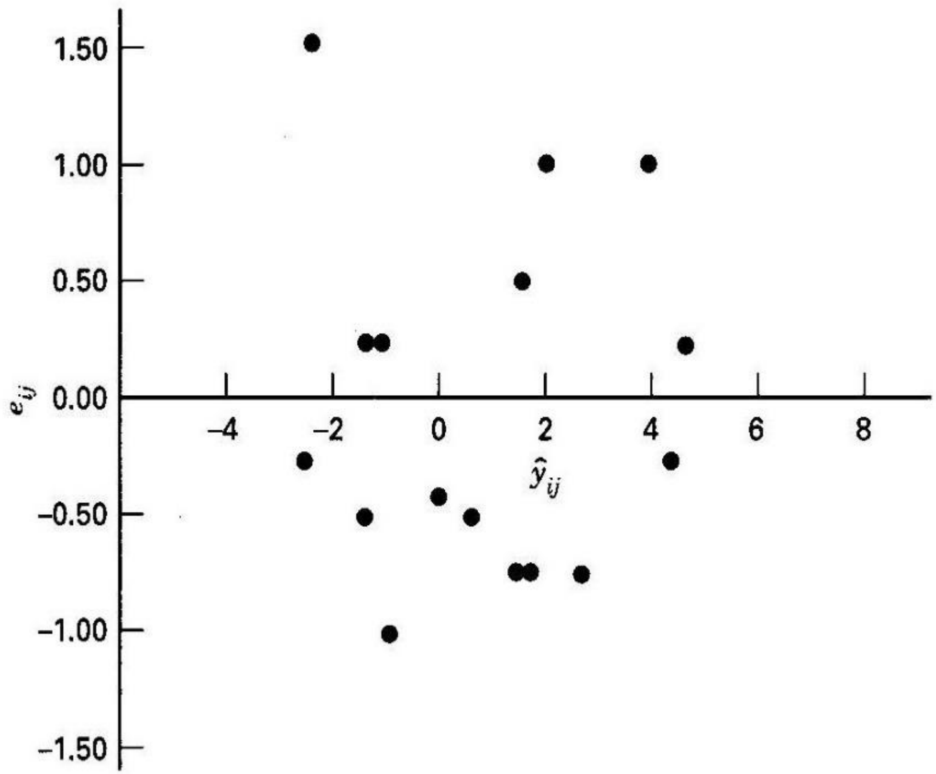

Figure 4-6 Plot of residuals versus $\hat{y}_{ij}$ for Example 4-1.

Choice of Sample Size

Choosing the **sample size**, or the **number of blocks** to run, is an important decision when using an RCBD. Increasing the number of blocks increases the number of replicates and the number of error degrees of freedom, making the design more sensitive. Any of the techniques discussed in Chapter 3 (Section 3-7) for selecting the number of replicates to run in a completely randomized single-factor experiment may be applied directly to the RCBD. For the case of a fixed factor, the operating characteristic curves in Appendix Chart V may be used with

$$\Phi^2 = \frac{b \sum_{i=1}^{a} \tau_i^2}{a\sigma^2} \tag{4-14}$$

where there are $a - 1$ numerator degrees of freedom and $(a - 1)(b - 1)$ denominator degrees of freedom.

EXAMPLE 4-2 ....

Consider the hardness testing problem described in Example 4-1. Suppose that we wish to determine the appropriate number of blocks to run if we are interested in detecting a true maximum difference in mean hardness readings of 0.4 with a high probability and a reasonable estimate of the standard deviation of the errors is $\sigma = 0.1$. (These values are given in the *original* units; recall that the analysis of variance was performed using *coded* data.) From Equation 3-49, the minimum value of Φ^2 is (writing b, the number of blocks, for n)

$$\Phi^2 = \frac{bD^2}{2a\sigma^2}$$

where D is the maximum difference we wish to detect. Thus,

$$\Phi^2 = \frac{b(0.4)^2}{2(4)(0.1)^2} = 2.0b$$

If we use $b = 3$ blocks, $\Phi = \sqrt{2.0b} = \sqrt{2.0(3)} = 2.45$, and there are $(a-1)(b-1) = 3(2) = 6$ error degrees of freedom. Appendix Chart V with $\nu_1 = a - 1 = 3$ and $\alpha = 0.05$ indicates that the β risk for this design is approximately 0.10 (power $= 1 - \beta = 0.90$). If we use $b = 4$ blocks, $\Phi = \sqrt{2.0b} = \sqrt{2.0(4)} = 2.83$, with $(a-1)(b-1) = 3(3) = 9$ error degrees of freedom, and the corresponding β risk is approximately 0.03 (power $= 1 - \beta = 0.97$). Either three or four blocks will result in a design having a high probability of detecting the difference between the mean hardness readings considered important. Because coupons (blocks) are inexpensive and readily available and the cost of testing is low, the experimenter decides to use four blocks.

Estimating Missing Values

When using the RCBD, sometimes an observation in one of the blocks is missing. This may happen because of carelessness or error or for reasons beyond our control, such as unavoidable damage to an experimental unit. A missing observation introduces a new problem into the analysis because treatments are no longer **orthogonal to blocks**; that

Table 4-7 Randomized Complete Block Design for the Hardness Testing Experiment with One Missing Value

	Coupon (Block)			
Type of Tip	1	2	3	4
1	−2	−1	1	5
2	−1	−2	x	4
3	−3	−1	0	2
4	2	1	5	7

is, every treatment does not occur in every block. There are two general approaches to the missing value problem. The first is an **approximate analysis** in which the missing observation is estimated and the usual analysis of variance is performed just as if the estimated observation were real data, with the error degrees of freedom reduced by 1. This approximate analysis is the subject of this section. The second is an **exact analysis**, which is discussed in Section 4-1.4.

Suppose the observation y_{ij} for treatment i in block j is missing. Denote the missing

Suppose the observation y_{ij} for treatment i in block j is missing. Denote the missing observation by x. As an illustration, suppose that in the hardness testing experiment of Example 4-1 coupon 3 was broken while tip 2 was being tested and that data point could not be obtained. The data might appear as in Table 4-7.

In general, we will let $y'_{..}$ represent the grand total with one missing observation, $y'_{i.}$ represent the total for the treatment with one missing observation, and $y'_{.j}$ be the total for the block with one missing observation. Suppose we wish to estimate the missing observation x so that x will have a minimum contribution to the error sum of squares. Because $SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$, this is equivalent to choosing x to minimize

$$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{1}{b}\sum_{i=1}^{a}\left(\sum_{j=1}^{b} y_{ij}\right)^2 - \frac{1}{a}\sum_{j=1}^{b}\left(\sum_{i=1}^{a} y_{ij}\right)^2 + \frac{1}{ab}\left(\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}\right)^2$$

or

$$SS_E = x^2 - \frac{1}{b}(y'_{i.} + x)^2 - \frac{1}{a}(y'_{.j} + x)^2 + \frac{1}{ab}(y'_{..} + x)^2 + R \qquad (4\text{-}15)$$

where R includes all terms not involving x. From $dSS_E/dx = 0$, we obtain

$$x = \frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)} \qquad (4\text{-}16)$$

as the estimate of the missing observation.

as the estimate of the missing observation.

For the data in Table 4-7, we find that $y'_{2.} = 1$, $y'_{.3} = 6$, and $y'_{..} = 17$. Therefore, from Equation 4-16,

$$x \equiv y_{23} = \frac{4(1) + 4(6) - 17}{(3)(3)} = 1.22$$

The usual analysis of variance may now be performed using $y_{23} = 1.22$ and reducing the error degrees of freedom by 1. The analysis of variance is shown in Table 4-8 on the facing page. Compare the results of this approximate analysis with the results obtained for the full data set (Table 4-5).

Table 4-8 Approximate Analysis of Variance for Example 4-1 with One Missing Value

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Type of tip	39.98	3	13.33	17.12	0.0008
Specimens (blocks)	79.53	3	26.51		
Error	6.22	8	0.78		
Total	125.73	14			

If several observations are missing, they may be estimated by writing the error sum of squares as a function of the missing values, differentiating with respect to each missing value, equating the results to zero, and solving the resulting equations. Alternatively, we may use Equation 4-16 iteratively to estimate the missing values. To illustrate the iterative approach, suppose that two values are missing. Arbitrarily estimate the first missing value, and then use this value along with the real data and Equation 4-16 to estimate the second. Now Equation 4-16 can be used to re-estimate the first missing value, and following this, the second can be re-estimated. This process is continued until convergence is obtained. In any missing value problem, the error degrees of freedom are reduced by one for each missing observation.

The Latin Square Design

- Text reference, Section 4.2, pg. 158
- These designs are used to simultaneously control (or eliminate) **two sources of nuisance variability**
- A significant assumption is that the three factors (treatments, nuisance factors) **do not interact**
- If this assumption is violated, the Latin square design will not produce valid results
- Latin squares are not used as much as the RCBD in industrial experimentation

The Rocket Propellant Problem – A Latin Square Design

TABLE 4.9
Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators				
	1	2	3	4	5
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

- This is a 5×5 Latin square design
- Statistical analysis?

Statistical Analysis of the Latin Square Design

- The statistical (effects) model is

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

- The statistical analysis (ANOVA) is much like the analysis for the RCBD.
- The analysis for the rocket propellant example follows

■ TABLE 4.10

Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{...}^2}{N}$	$p - 1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$	$p - 1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N}$	$p - 1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$	$p^2 - 1$		

■ **TABLE 4.12**
Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

As in any design problem, the experimenter should investigate the adequacy of the model by inspecting and plotting the residuals. For a Latin square, the residuals are given by

$$\begin{aligned} e_{ijk} &= y_{ijk} - \hat{y}_{ijk} \\ &= y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}_{...} \end{aligned}$$

Occasionally, one observation in a Latin square is missing. For a $p \times p$ Latin square, the missing value may be estimated by

$$y_{ijk} = \frac{p(y'_{i..} + y'_{.j.} + y'_{..k}) - 2y'_{...}}{(p-2)(p-1)} \tag{4-24}$$

where the primes indicate totals for the row, column, and treatment with the missing value, and $y'_{...}$ is the grand total with the missing value.

Latin squares can be useful in situations where the rows and columns represent factors the experimenter actually wishes to study and where there are no randomization restrictions. Thus, three factors (rows, columns, and letters), each at p levels, can be investigated in only p^2 runs. This design assumes that there is no interaction between the factors. More will be said later on the subject of interaction.

4-4 BALANCED INCOMPLETE BLOCK DESIGNS

In certain experiments using randomized block designs, we may not be able to run all the treatment combinations in each block. Situations like this usually occur because of shortages of experimental apparatus or facilities or the physical size of the block. For example, in the hardness testing experiment (Example 4-1), suppose that because of their size each coupon can be used only for testing three tips. Therefore, each tip cannot be tested on each coupon. For this type of problem it is possible to use randomized block designs in which every treatment is not present in every block. These designs are known as **randomized incomplete block designs**.

When all treatment comparisons are equally important, the treatment combinations used in each block should be selected in a balanced manner, that is, so that any pair of treatments occur together the same number of times as any other pair. Thus, a **balanced incomplete block design (BIBD)** is an incomplete block design in which any two treatments appear together an equal number of times. Suppose that there are a treatments and that each block can hold exactly k ($k < a$) treatments. A balanced incomplete block design may be constructed by taking $\binom{a}{k}$ blocks and assigning a different combination of treatments to each block. Frequently, however, balance can be obtained with fewer than $\binom{a}{k}$ blocks. Tables of BIBDs are given in Fisher and Yates (1953), Davies (1956), and Cochran and Cox (1957).

As an example, suppose that a chemical engineer thinks that the time of reaction for a chemical process is a function of the type of catalyst employed. Four catalysts are currently being investigated. The experimental procedure consists of selecting a batch of raw material, loading the pilot plant, applying each catalyst in a separate run of the pilot plant, and observing the reaction time. Because variations in the batches of raw material may affect the performance of the catalysts, the engineer decides to use batches of raw material as blocks. However, each batch is only large enough to permit three catalysts

Table 4-22 Balanced Incomplete Block Design for Catalyst Experiment

Teratment (Catalyst)	Block (Batch of Raw Material) 1	2	3	4	$y_{i.}$
1	73	74	—	71	218
2	—	75	67	72	214
3	73	75	68	—	216
4	75	—	72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

to be run. Therefore, a randomized incomplete block design must be used. The balanced incomplete block design for this experiment, along with the observations recorded, is shown in Table 4-22 at the bottom of the previous page. The order in which the catalysts are run in each block is randomized.

4-4.1 Statistical Analysis of the BIBD

As usual, we assume that there are a treatments and b blocks. In addition, we assume that each block contains k treatments, that each treatment occurs r times in the design (or is replicated r times), and that there are $N = ar = bk$ total observations. Furthermore, the number of times each pair of treatments appears in the same block is

$$\lambda = \frac{r(k-1)}{a-1}$$

If $a = b$, the design is said to be **symmetric**.

The parameter λ must be an integer. To derive the relationship for λ, consider any treatment, say treatment 1. Because treatment 1 appears in r blocks and there are $k - 1$ other treatments in each of those blocks, there are $r(k - 1)$ observations in a block containing treatment 1. These $r(k - 1)$ observations also have to represent the remaining $a - 1$ treatments λ times. Therefore, $\lambda(a - 1) = r(k - 1)$.

The **statistical model** for the BIBD is

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \tag{4-26}$$

where y_{ij} is the ith observation in the jth block, μ is the overall mean, τ_i is the effect of the ith treatment, β_j is the effect of the jth block, and ϵ_{ij} is the NID $(0, \sigma^2)$ random error component. The total variability in the data is expressed by the total corrected sum of squares:

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{N} \tag{4-27}$$

Total variability may be partitioned into

$$SS_T = SS_{\text{Treatments(adjusted)}} + SS_{\text{Blocks}} + SS_E$$

where the sum of squares for treatments is **adjusted** to separate the treatment and the block effects. This adjustment is necessary because each treatment is represented in a different set of r blocks. Thus, differences between unadjusted treatment totals $y_{1.}$, $y_{2.}, \ldots, y_{a.}$ are also affected by differences between blocks.

The block sum of squares is

$$SS_{\text{Blocks}} = \frac{1}{k}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N} \tag{4-28}$$

where $y_{.j}$ is the total in the jth block. SS_{Blocks} has $b - 1$ degrees of freedom. The adjusted treatment sum of squares is

$$SS_{\text{Treatments(adjusted)}} = \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda a} \tag{4-29}$$

Table 4-23 Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\dfrac{k\sum Q_i^2}{\lambda a}$	$a-1$	$\dfrac{SS_{\text{Treatments(adjusted)}}}{a-1}$	$F_0 = \dfrac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\dfrac{1}{k}\sum y_{.j}^2 - \dfrac{y_{..}^2}{N}$	$b-1$	$\dfrac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$N-a-b+1$	$\dfrac{SS_E}{N-a-b+1}$	
Total	$\sum\sum y_{ij}^2 - \dfrac{y_{..}^2}{N}$	$N-1$		

where Q_i is the adjusted total for the ith treatment, which is computed as

$$Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij}y_{.j} \qquad i = 1, 2, \ldots, a \tag{4-30}$$

with $n_{ij} = 1$ if treatment i appears in block j and $n_{ij} = 0$ otherwise. The adjusted treatment totals will always sum to zero. $SS_{\text{Treatments(adjusted)}}$ has $a - 1$ degrees of freedom. The error sum of squares is computed by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments(adjusted)}} - SS_{\text{Blocks}} \tag{4-31}$$

and has $N - a - b + 1$ degrees of freedom.

The appropriate statistic for testing the equality of the treatment effects is

$$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$$

The analysis of variance is summarized in Table 4-23.

EXAMPLE 4-5 ....

Consider the data in Table 4-22 for the catalyst experiment. This is a BIBD with $a = 4$, $b = 4$, $k = 3$, $r = 3$, $\lambda = 2$, and $N = 12$. The analysis of this data is as follows. The total sum of squares is

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{12}$$

$$= 63{,}156 - \frac{(870)^2}{12} = 81.00$$

The block sum of squares is found from Equation 4-28 as

$$SS_{\text{Blocks}} = \frac{1}{3} \sum_{j=1}^{4} y_{.j}^2 - \frac{y_{..}^2}{12}$$

$$= \frac{1}{3} [(221)^2 + (207)^2 + (224)^2 + (218)^2] - \frac{(870)^2}{12} = 55.00$$

Table 4-24 Analysis of Variance for Example 4-5

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Treatments (adjusted for blocks)	22.75	3	7.58	11.66	0.0107
Blocks	55.00	3	—		
Error	3.25	5	0.65		
Total	81.00	11			

To compute the treatment sum of squares adjusted for blocks, we first determine the adjusted treatment totals using Equation 4-30 as

$$Q_1 = (218) - \tfrac{1}{3}(221 + 224 + 218) = -9/3$$
$$Q_2 = (214) - \tfrac{1}{3}(207 + 224 + 218) = -7/3$$
$$Q_3 = (216) - \tfrac{1}{3}(221 + 207 + 224) = -4/3$$
$$Q_4 = (222) - \tfrac{1}{3}(221 + 207 + 218) = 20/3$$

The adjusted sum of squares for treatments is computed from Equation 4-29 as

$$SS_{\text{Treatments(adjusted)}} = \frac{k \sum_{i=1}^{4} Q_i^2}{\lambda a} = \frac{3[(-9/3)^2 + (-7/3)^2 + (-4/3)^2 + (20/3)^2]}{(2)(4)} = 22.75$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments(adjusted)}} - SS_{\text{Blocks}} = 81.00 - 22.75 - 55.00 = 3.25$$

The analysis of variance is shown in Table 4-24. Because the *P*-value is small, we conclude that the catalyst employed has a significant effect on the time of reaction.

If the factor under study is fixed, tests on individual treatment means may be of interest. If orthogonal contrasts are employed, the contrasts must be made on the **adjusted treatment totals**, the $\{Q_i\}$ rather than the $\{y_{i.}\}$. The contrast sum of squares is

$$SS_C = \frac{k\left(\sum_{i=1}^{a} c_i Q_i\right)^2}{\lambda a \sum_{i=1}^{a} c_i^2}$$

where $\{c_i\}$ are the contrast coefficients. Other multiple comparison methods may be used to compare all the pairs of adjusted treatment effects, which we will find in Section

4-4.2, are estimated by $\hat{\tau}_i = kQ_i/(\lambda a)$. The standard error of an adjusted treatment effect is

$$S = \sqrt{\frac{kMS_E}{\lambda a}} \tag{4-32}$$

In the analysis that we have described the total sum of squares has been partitioned into an adjusted sum of squares for treatments, an unadjusted sum of squares for blocks, and an error sum of squares. Sometimes we would like to assess the block effects. To do this, we require an alternate partitioning of SS_T, that is,

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks(adjusted)}} + SS_E$$

Here $SS_{\text{Treatments}}$ is unadjusted. If the design is symmetric, that is, if $a = b$, a simple formula may be obtained for $SS_{\text{Blocks(adjusted)}}$. The adjusted block totals are

$$Q'_j = y_{.j} - \frac{1}{r}\sum_{i=1}^{a} n_{ij} y_{i.} \qquad j = 1, 2, \ldots, b \tag{4-33}$$

and

$$SS_{\text{Blocks(adjusted)}} = \frac{r\sum_{j=1}^{b} (Q'_j)^2}{\lambda b} \tag{4-34}$$

The BIBD in Example 4-5 is symmetric because $a = b = 4$. Therefore,

$$Q'_1 = (221) - \tfrac{1}{3}(218 + 216 + 222) = 7/3$$

$$Q'_2 = (224) - \tfrac{1}{3}(218 + 214 + 216) = 24/3$$

$$Q'_3 = (207) - \tfrac{1}{3}(214 + 216 + 222) = -31/3$$

$$Q'_4 = (218) - \tfrac{1}{3}(218 + 214 + 222) = 0$$

and

$$SS_{\text{Blocks(adjusted)}} = \frac{3[(7/3)^2 + (24/3)^2 + (-31/3)^2 + (0)^2]}{(2)(4)} = 66.08$$

Also,

$$SS_{\text{Treatments}} = \frac{(218)^2 + (214)^2 + (216)^2 + (222)^2}{3} - \frac{(870)^2}{12} = 11.67$$

Table 4-25 Analysis of Variance for Example 4-5, Including Both Treatments and Blocks

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Treatments (adjusted)	22.75	3	7.58	11.66	0.0107
Treatments (unadjusted)	11.67	3			
Blocks (unadjusted)	55.00	3			
Blocks (adjusted)	66.08	3	22.03	33.90	0.0010
Error	3.25	5	0.65		
Total	81.00	11			