Chapter 4- Part 3: TIME SERIES METHODS (Trend and Seasonal processes)

5.2. Trend Process

Sometimes scatter plot of data shows specific type of trend. Examining this data clearly indicates that the underlying process is not constant but is steadily increasing or decreasing.

To accurately forecast this time series, a model that incorporates trend is necessary. The underlying model for a process with linear trend is given by

$$d_t = \alpha + bt + \varepsilon_t$$

In which b is the slope of the trend and other notation is as previously defined. If b is positive, the process is increasing over time. But a negative b implies a decreasing process. We will discuss an increasing trend, but the methodology is also applicable to a decreasing trend.

To forecast when trend is present, we need to estimate the constant and the slope; there are many ways to do so, including regression and variations on moving averages and exponential smoothing. Here we discuss a modification of simple exponential smoothing for trend.

5.2.1. Double Exponential Smoothing

If we were to forecast a trend model using simple exponential smoothing, the forecast would be late reacting to the growth. Therefore, the forecast would tend to underestimate the actual demand. To correct this underestimate, we could estimate the slope and multiply this estimate by the number of periods in the future we wish to forecast.

Double exponential smoothing method is the extension of simple exponential smoothing method for using in trended time series, and in a particular, in situations when there is a trend in data. This method uses two smoothing parameters to update the level and trend components. In this forecasting process three equations are used: the first equation is for smoothing time series, the second equation is for smoothing trend, and the third equation is for the combination of above two equations. So we have:

$$S_t = \alpha d_t + (1-\alpha)(S_{t-1} + b_{t-1}) \qquad 0 < \alpha < 1$$

$$b_t = \beta(S_t - S_{t-1}) + (1-\beta)b_{t-1} \qquad 0 < \beta < 1$$

$$F_{t,t+1} = S_t + b_t$$

All the characters used in a simple exponential smoothing equation represent the same meaning in a double exponential smoothing equation appearing (while α is smoothing constant), but β is a trend smoothing constant. S_t is the smoothed constant process value for the period t, and b_t is the smoothed trend value for the period t.

Example: suppose we have the actual sales data for 12 months represented in table. Plot data and forecast demand of each period by double exponential smoothing method. Forecast sales of period 13 by double exponential smoothing method.

Month	Sales	S_t	b_t	F_t
1	150			
2	162	150	12	150
3	159	161.1	11.55	162
4	178	174.255	12.3525	172.65
5	195	189.1253	13.61138	186.6075
6	219	207.6156	16.05088	202.7367
7	200	216.56653	12.5009	223.6665
8	253	236.2472	16.090785	229.0674
9	300	266.6366	23.240092	252.338
10	286	288.7137	22.658596	289.8767
11	319	313.6606	23.802748	311.3723
12	332	335.8243	22.98324	337.4633

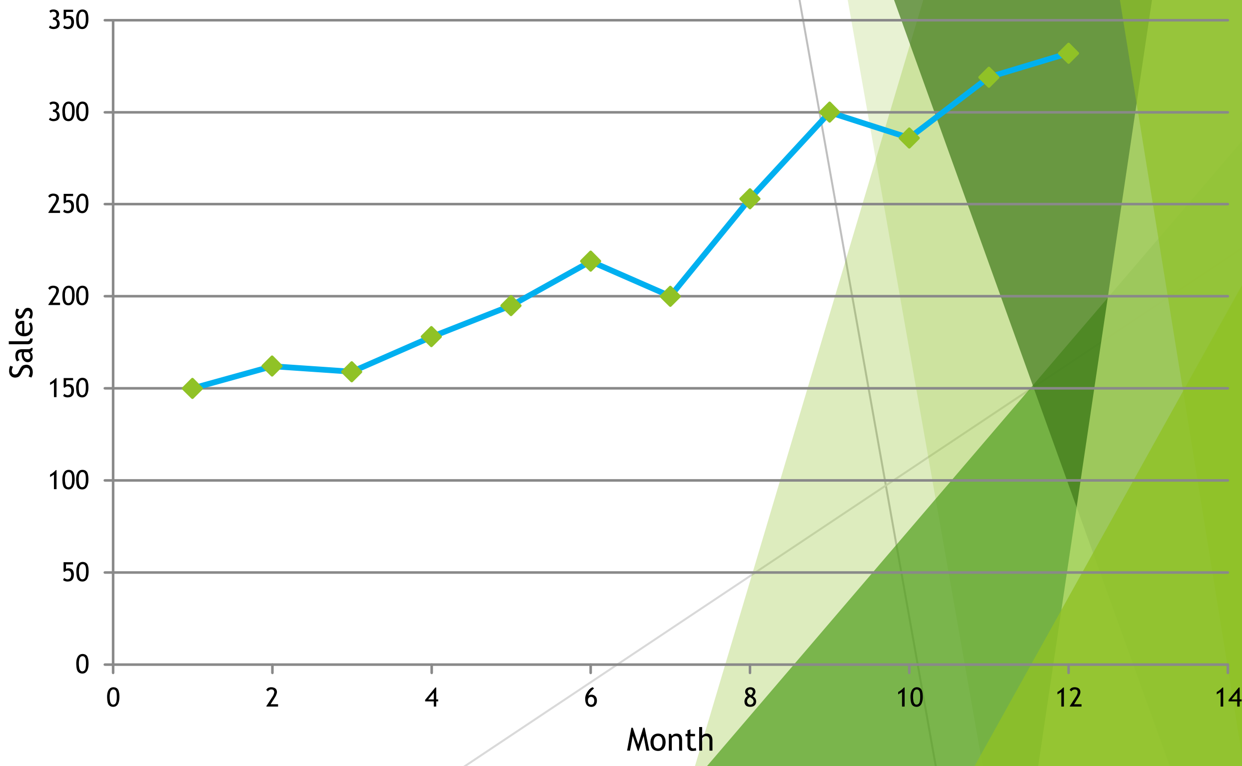


Figure 4.10. Actual sales

Now, we need to calculate the forecast for the period 13. We find the sum of S_{12} and b_{12}.

$$F_{t,t+1} = S_t + b_t$$

$$F_{12,13} = 335.8243 + 22.98324 = 358.8076$$

Note that the time series exhibits a growing trend, and then must use the double exponential smoothing. Firstly the initial values for S and b are determined. S_1 and b_1 are not defined, and one way to identify these values is assuming that the initial value is equal to its expectations. Using this as starting point, set $S_2 = d_1 = 150$. Then subtract d_1 from d_2 to get b_2:$b_2 = d_2 - d_1 = 162 - 150 = 12$. Hence, at the end of period 2, forecast for the period 3 is $162(F_3 = 150 + 12)$. We choose $\alpha = 0.3\ and\ \beta = 0.5$ to solve this example. The actual sale for the period 3 was 159.

$$S_3 = 0.3 * 159 + (1 - 0.3)(150 + 12) = 161.1$$

$$b_3 = 0.5(161.1 - 150) + (1 - 0.5)12 = 11.55$$

$$F_t = 161.1 + 11.55 = 172.65$$

Simple way:

To implement double exponential smoothing at time t, we need values of S_{t-1} and b_{t-1}.

$$S_t = \alpha d_t + (1 - \alpha)(S_{t-1} + b_{t-1}) \qquad 0 < \alpha < 1$$

$$b_t = \beta(S_t - S_{t-1}) + (1 - \beta)b_{t-1} \qquad 0 < \beta < 1$$

$$F_{t,t+k} = S_t + kb_t$$

A simple way to do this is first divide the data into two equal groups, and compute the average of each. To convert this difference to an estimate of the slope, divide it by the number of time periods separating the two average. Then to get an estimate of the intercept, use the overall average plus the slope estimate per period times the number of periods from the midpoint to the current period.

It means that $b_t = \frac{\bar{d}_2 - \bar{d}_1}{n/2}$ and $S_t = \bar{d} + \frac{n-1}{2} b_t$

Table 4.11 gives demand data for computer paper.

Develop a forecast of computer paper sales for months 25 and 30.

Assume that current time is 8, forecast of computer paper sales for months 25 and 30

Table 4.11. Computer paper sales (in cases)

Month	Sales	Month	Sales	Month	Sales
1	116	9	163	17	210
2	133	10	163	18	207
3	139	11	164	19	225
4	157	12	191	20	223
5	154	13	201	21	257
6	159	14	219	22	232
7	162	15	207	23	240
8	172	16	205	24	241

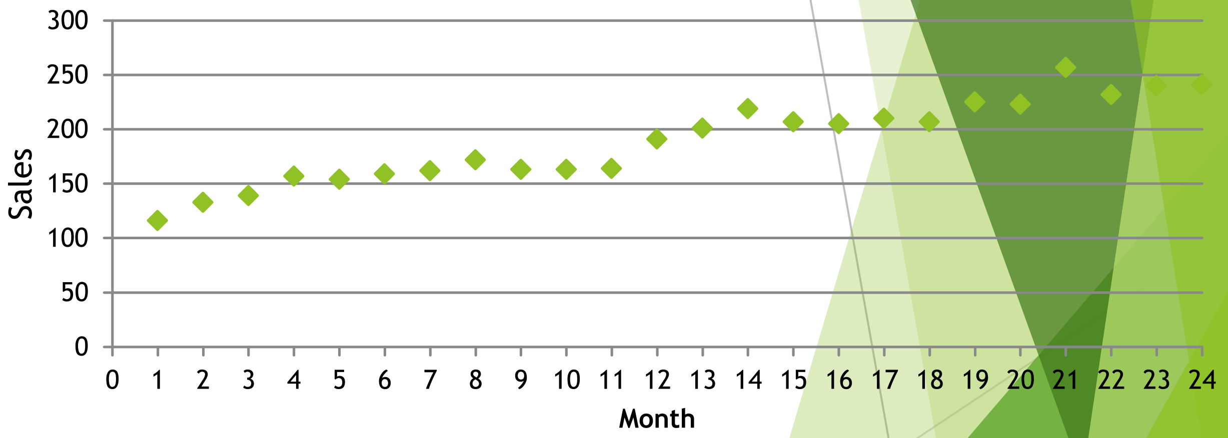


Figure 4.11. Computer paper sales

Solution:

First, compute the averages of the months 1 to 12, and 13 to 24.

$$\bar{d}_1 = \frac{116 + 133 + 139 + 157 + 154 + 159 + 162 + 172 + 163 + 163 + 164 + 191}{12} = 156.08$$

$$\bar{d}_2 = \frac{201 + 219 + 207 + 205 + 210 + 207 + 225 + 223 + 257 + 232 + 240 + 241}{12} = 222.25$$

$$\bar{d} = \frac{116 + 133 + 139 + \cdots + 232 + 240 + 241}{24} = 189.16$$

$$b_t = \frac{\bar{d}_2 - \bar{d}_1}{n/2} \rightarrow b_{24} = \frac{222.25 - 156.08}{12} = 5.51$$

$$S_t = \bar{d} + \frac{n-1}{2} b_t \rightarrow S_{24} = 189.16 + \frac{24-1}{2}(5.51) = 252.53$$

Once we have our initial values, we can forecast for future periods. The forecast for period 25 is

$$F_{24,25} = S_{24} + (1 * b_{24}) = 252.53 + (1 * 5.51) = 258.04$$

Similarly, forecasting for period 30 gives

$$F_{24,30} = S_{24} + (6 * b_{24}) = 252.53 + (6 * 5.51) = 285.59$$

Second part: First, compute the averages of the months 1 to 4, and 5 to 8.

$$\bar{d}_1 = \frac{116 + 133 + 139 + 157}{4} = 136.25$$

$$\bar{d}_2 = \frac{154 + 159 + 162 + 172}{4} = 161.75$$

$$\bar{d} = \frac{116 + 133 + 139 + 157 + 154 + 159 + 162 + 172}{8} = 149$$

$$b_t = \frac{\bar{d}_2 - \bar{d}_1}{n/2} \rightarrow b_8 = \frac{161.75 - 136.25}{4} = 6.375$$

$$S_t = \bar{d} + \frac{n-1}{2} b_t \rightarrow S_8 = 149 + \frac{8-1}{2}(6.375) = 171.3125$$

Once we have our initial values, we can forecast for future periods. The forecast for period 25 is

$$F_{t,t+k} = S_t + kb_t$$

$$F_{8,25} = S_8 + (17 * b_8) = 171.3125 + (17 * 6.375) = 279.6875$$

Similarly, forecasting for period 30 gives

$$F_{8,30} = S_8 + 22 * b_8 = 171.3125 + (22 * 6.375) = 311.5625$$

5.3. Seasonal Process

Outdoor Furniture makes swing, people typically buy more swings in the warmer months than they do in the cooler months, so sales change with the seasons. Suppose outdoor furniture's swings are very good, and word-of-mouth advertising is increasing the number of people who buy them. Their data, which reflect seasonality and trend is given in table 4-12 and plotted in Figure 4-12.

Table 4-12: Quarterly sales of Outdoor Furniture's swings

	Year		
Quarter	1	2	3
1	60	69	84
2	234	266	310
3	163	188	212
4	50	59	64
Yearly average	126.75	145.50	167.50
Overall average			146.58

Here, a year can be divided into four seasons, each three months or a quarter of a year. Many processes naturally have different number of seasons in a year. If the time periods are weeks, the year would have 52 seasons. Periods of months and quarters have 12 and 4 seasons in a year, respectively. Other processes may have a season that is not based on years, but there should be some underlying explanation for the seasonality. The methods presented here can be used for any season length.

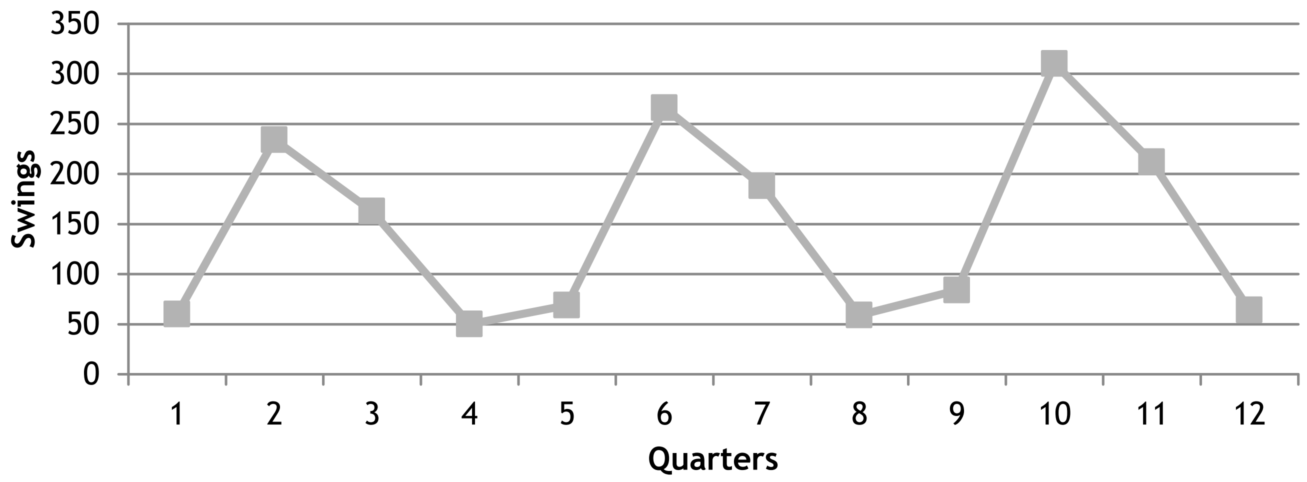


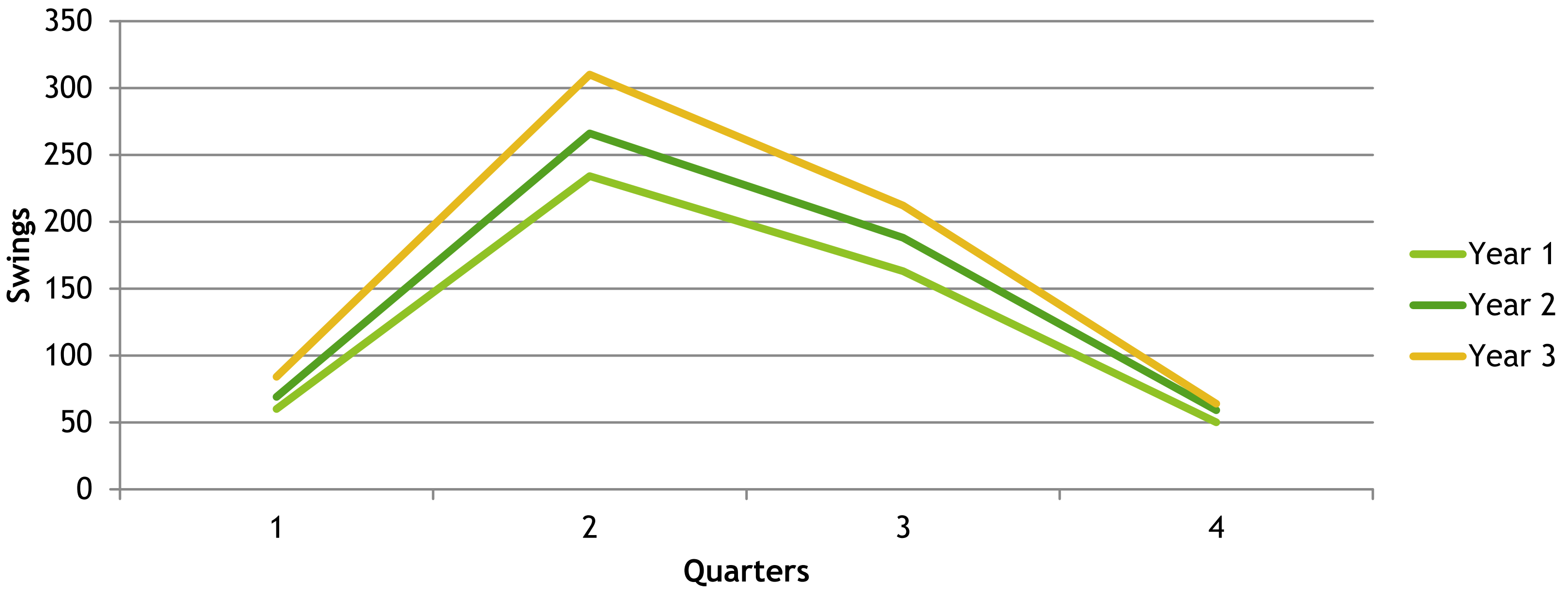
Figure 4-12 Seasonal data with trend

To highlight the seasonality and trend, we plot by season in figure 4-13. Demand for each quarter of the first year is below the demand of the same quarter for the second year, and the second year value is below the third year value, so demand appears to be growing. A good model must consider the constant portion of demand, the trend, and seasonality.

Several models consider all three factors; we will discuss a popular multiplicative model proposed by Winters (1960). Formally, the model is:

$$d_t = (a + bt)c_t + \varepsilon_t$$

Where a= the constant portion, b= the slope of the trend component, c_t= the seasonal factor for period t, ε_t= the uncontrollable randomness.



The forecasting method is to estimate the parameters of the model and use them to generate the forecast. We estimate the constant component independently of the trend and seasonal factors, so we call it the depersonalized constant. Likewise, the trend factor should be independent of the seasonal factors. The seasonal factors can be viewed as a percentage of the constant and trend components for period t; if demand in a particular period of the season is lower than the constant/trend component, the seasonal factor will be less than one, and if demand is higher, it will be greater than one. The seasonal factors must sum to the number of seasons per year. To forecast, we get initial estimates of the components of the model and update them using exponential smoothing.

Let: d_t = the demand in period t

L = length of season in a year (or other time frame)

T = the number of periods of data available

$T = mL$ where

m = the number of full seasons of data available

S_t = the estimate for the constant term a calculated at period t

b_t = the estimate for the trend term b calculated at time t

C_t = the estimate of the seasonal component for period t

To start the procedure, we need an initial value for S_t. A naïve estimate is an average of one or more complete seasons of data. Partial seasons should not be used; using only the first 9 data points might give a poor estimate because larger or smaller demand in the first quarter does not reflect "average" demand. With trend, averaging one or more complete years of historical data values does not give an initial estate of a. This average includes the "lower" demand at the beginning as well as the "higher" demand at the end of the historical data.

Winter's Method:

Step 1: Calculate the average of each of the last two seasons of data, $\overline{d}_1$ and $\overline{d}_2$, then

$b_t = \frac{\overline{d}_2 - \overline{d}_1}{L}$ $\qquad L = length\ of\ season$

Step 2: calculate the overall average

$$\overline{d} = \frac{1}{t}\sum_{t=1}^{n} d_t$$

And use it and b_t to calculate the estimate constant S_t

$$S_t = \overline{d} + \left(\frac{n-1}{2}\right)b_t$$

Step 3: Calculate the seasonal factors using the equation

$$C_t = \frac{1}{m}\sum_{i=0}^{m-1} \frac{d_{t-iL}}{S_t - (T - [t - iL])b_t} \qquad t = T - L + 1, T - L + 2, \ldots, T$$

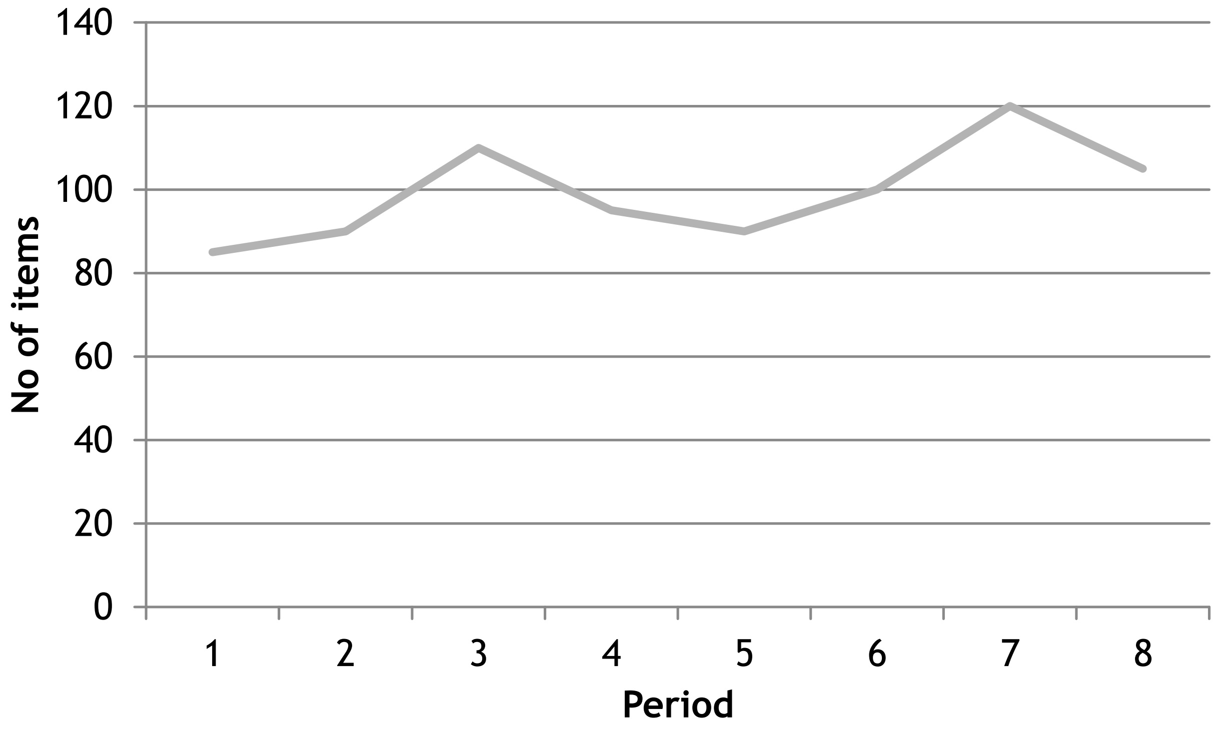
Step 4: To forecast *k* periods ahead, use

$F_{t,t+k} = (S_t + kb_t)C_{t+k-gl},$

g = the smallest integer greater than or equal to *k/L*

Example: data of one specific seasonal product represents in this table. Apply winter method and forecast demands for periods of 9, 10, 11, and 12

Period	1	2	3	4	5	6	7	8
d_t	85	90	110	95	90	100	120	105



Step 1:

Period	d_t
(1 W) 1	85
(1 Sp) 2	90
(1 Su) 3	110
(1 F) 4	95
(2 W) 5	90
(2 Sp) 6	100
(2 Su) 7	120
(2 F) 8	105

$$\bar{d}_1 = \frac{85+90+110+95}{4} = 95$$

$$\bar{d}_2 = \frac{90+100+120+105}{4} = 103.75$$

$$b_t = \frac{\bar{d}_2 - \bar{d}_1}{L} = \frac{103.75 - 95}{4} = 2.1875$$

Step 2:

$$\bar{d} = \frac{1}{t}\sum_{t=1}^{n} d_t = \frac{85+90+110+95+90+100+120+105}{8} = 99.38$$

$$S_t = \bar{d} + \left(\frac{n-1}{2}\right) b_t = 99.38 + \left(\frac{8-1}{2}\right) 2.1875 = 107.03125$$

Step 3:

$$C_t = \frac{1}{m}\sum_{i=0}^{m-1}\frac{d_{t-iL}}{S_t-(T-[t-iL])b_t} \qquad t = T-L+1, T-L+2, \dots, T$$

$$C_5 = \frac{1}{2}\left(\frac{90}{107.3125-3(2.1875)}+\frac{85}{107.3125-7(2.1875)}\right) = 0.90861$$

$$C_6 = \frac{1}{2}\left(\frac{100}{107.3125-2(2.1875)}+\frac{90}{107.3125-6(2.1875)}\right) = 0.96350$$

$$C_7 = \frac{1}{2}\left(\frac{120}{107.3125-1(2.1875)}+\frac{110}{107.3125-5(2.1875)}\right) = 1.14144$$

$$C_8 = \frac{1}{2}\left(\frac{105}{107.3125-0(2.1875)}+\frac{95}{107.3125-4(2.1875)}\right) = 0.97115$$

Step 4:

$$F_{t,t+k} = (S_t + kb_t)C_{t+k-gL}$$

$$F_{8,9} = (S_8 + 1b_8)C_5 = (107.03125 + 1 * 2.1875)0.90861 = 99.24$$

$$F_{8,10} = (S_8 + 2b_8)C_6 = (107.03125 + 2 * 2.1875)0.96350 = 107.34$$

$$F_{8,11} = (S_8 + 3b_8)C_7 = (107.03125 + 3 * 2.1875)1.14144 = 129.66$$

$$F_{8,12} = (S_8 + 4b_8)C_8 = (107.03125 + 4 * 2.1875)0.97115 = 112.44$$