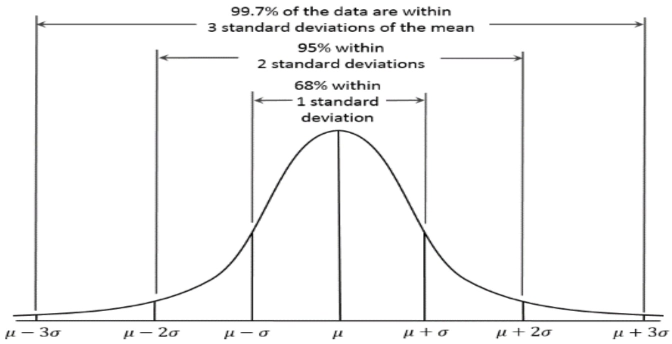
Normal Distribution

If Probability Density Function of a continuous random variable like X with parameters μ and σ^2 is:

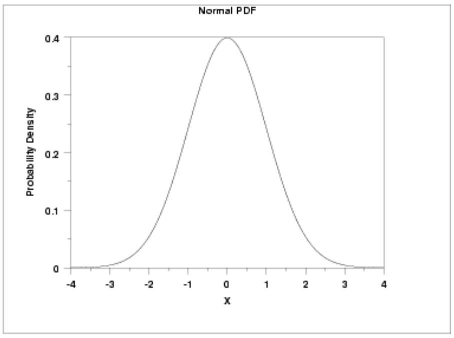
$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < +\infty$$

We can say that variable is normally distributed
parameters μ and σ^2 are mean and variance of random variable



Standard Normal Distribution
The distribution of normal random variable with mean 0 and variance 1 is called a standard normal distribution (Z).

$$Z = \frac{X-\mu}{\sigma} \quad \rightarrow \quad f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} \qquad -\infty < Z < +\infty$$

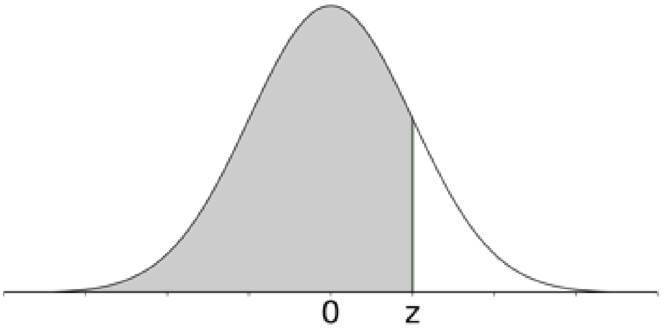


A **standard normal table**, also called the **unit normal table** or **Z table**, is a mathematical table for the values of Φ, which are the values of the cumulative distribution function of the normal distribution. It is used to find the probability that a statistic is observed below, above, or between values on the standard normal distribution, and by extension, any normal distribution. Since probability tables cannot be printed for every normal distribution, as there are an infinite variety of normal distributions, it is common practice to convert a normal to a standard normal and then use the standard normal table to find probabilities .

$$p(Z < z) = \varphi(z)$$

$$p(Z > z) = 1 - p(Z < z)$$

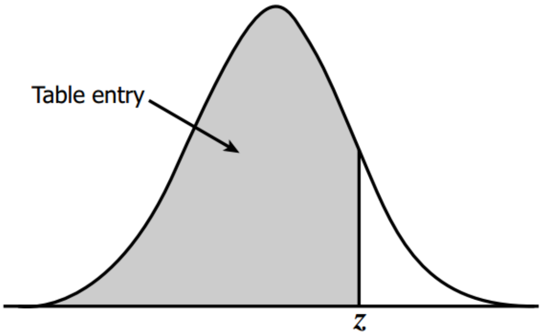
Table of Standard Normal Probabilities for Positive Z-Scores



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441

$$p(z < 1.00) = \varphi(1) = 0.8413$$

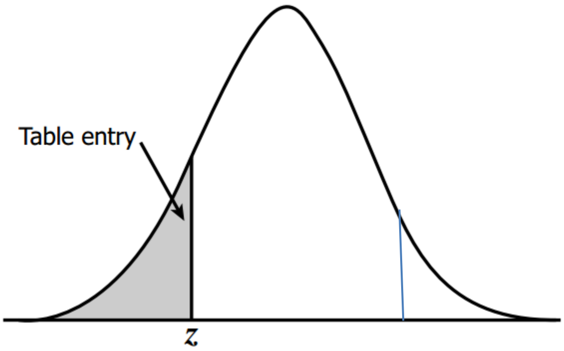
$$p(z > 1.25) = 1 - p(z < 1.25) = 1 - \varphi(1.25) = 1 - 0.8944 = 0.1056$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$$p(z \leq 2.20) = \varphi(2.2) = 0.9861$$

$$p(z > 0.25) = 1 - p(z < 0.25) = 1 - \varphi(0.25) = 1 - 0.5987 = 0.4013$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

$$p(z \leq -0.64) = \varphi(-0.64) = 0.2611$$

$$p(z > -1.95) = 1 - p(z < -1.95) = 1 - \varphi(-1.95) = 1 - 0.0256 = 0.9744$$

$$p(z > -3.55) = 1 - p(z < -3.55) = 1 - 0.0000 \cong 1$$

Example: a certain type of storage battery lasts, on average, 3 years with standard deviation of 0.5 year. Assume that the battery lives are normally distributed. Find

- Probability that a given battery will last less than 2.3 years
- Probability that a given battery will last more than or equal 3.3 years

$$p(x < 2.3) = p\left(\frac{x-\mu}{\sigma} < \frac{2.3-3}{0.5}\right) = p(z < -1.4) = \varphi(-1.4) = 0.0808$$

$$p(x > 3.3) = 1 - p(x < 3.3) = 1 - p\left(\frac{x-\mu}{\sigma} < \frac{3.3-3}{0.5}\right) = 1 - p(z < 0.6)$$
$$= 1 - 0.7257 = 0.2743$$

Example: an electrical firm produces light bulbs that their lives before burn-out, are normally distributed with mean $\mu = 800\ hrs$ and variance $\sigma^2 = 1600\ hrs$. Find

- Probability that a bulb burn between 778 and 834 hours

$$p(778 < x < 834) = p\left(\frac{778-800}{40} < \frac{x-\mu}{\sigma} < \frac{834-800}{40}\right)$$
$$= p(-0.55 < z < 0.85) = p(z < 0.85) - p(z < -0.55)$$
$$= 0.8023 - 0.2912 = 0.5114$$

Example: an electrical firm produces two type of light bulbs that their lives before burn-out, are normally distributed with mean $\mu = 800\ \&\ 700\ hrs$ and variance $\sigma^2 = 1600\ \&\ 1400\ hrs$. Find

- Probability that total life of two bulbs will exceed 1700 hours
- Probability that difference life of two bulbs will not exceed 50 hours

$$p(x + y > 1700) = \left(\frac{(x+y) - \mu_{x+y}}{\sigma_{x+y}} > \frac{1700-1500}{\sqrt{3000}}\right) = p\left(z > \frac{200}{54.77}\right)$$
$$= p(z > 3.65) = 1 - p(z < 3.65) = 1 - 1 = 0$$

$$\mu_{x+y} = \mu_x + \mu_y = 800 + 700 = 1500$$
$$\sigma^2{}_{x+y} = \sigma^2{}_x + \sigma^2{}_y = 1600 + 1400 = 3000$$

$$p(x - y < 50) = \left(\frac{(x-y) - \mu_{x-y}}{\sigma_{x-y}} < \frac{50-100}{\sqrt{3000}}\right) = p\left(z < \frac{050}{54.77}\right)$$
$$= p(z < -0.91) = 0.1814$$

$$\mu_{x-y} = \mu_x - \mu_y = 800 - 700 = 100$$
$$\sigma^2{}_{x-y} = \sigma^2{}_x + \sigma^2{}_y = 1600 + 1400 = 3000$$

Central Limit Theorem:

The sum of large number of independent random variables is approximately normally distributed.

Let $x_1, x_2, \dots, x_n$ be a sample from a population having $mean = \mu$ and $sd = \sigma$.

For n large, the sum of $x_1 + x_2 + \cdots + x_n$ will approximately have a normal distribution with mean $n\mu$ and $sd = \sigma\sqrt{n}$.

Example: An assurance company has 10,000 automobile policy holders, if the expected yearly claim per policy holder is 260$ with standard deviation of 800$, approximate the probability that total yearly claim exceeds 2.8$ million.

$$y = \sum_{i=1}^{10000} x_i \sim N(n\mu, \sigma\sqrt{n}) \sim N(260 * 10^4, 8 \times 10^4)$$

$$p(y > 2.8 \times 10^6) = p\left(\frac{y - \bar{y}}{\sigma_y} > \frac{2.8 \times 10^6 - 2.6 \times 10^6}{8 \times 10^4}\right) = p(z > 2.5) = 1 - p(z < 2.5)$$
$$= 1 - 0.9938 = 0.0062$$

When n is large $(n > 30)$ **then** $\bar{x} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$

Example: The blood cholesterol level of a population of workers has mean equal 202 and standard deviation equal 14. If a sample of 36 workers is selected, approximate the probability that sample mean of their blood cholesterol level will be between 198 and 206.

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \sim N(202, \frac{14}{\sqrt{36}})$$

$$p(198 < \bar{x} < 206) = p\left(\frac{198 - 202}{14/\sqrt{36}} < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{206 - 202}{14/\sqrt{36}}\right) = p(-1.71 < z < 1.71)$$
$$= p(z < 1.71) - p(z < -1.71) = 0.9564 - 0.0436 = 0.9128$$

Note 1: for central limit theorem $n \geq 30$ is enough

Note 2 : if the underlying population distribution is normal, then the sample mean $\bar{x}$ will also be normal. No matter what the sample size is.