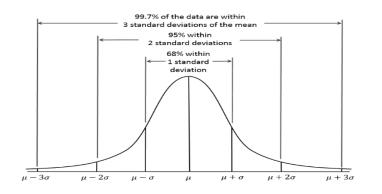
Normal Distribution

If Probability Density Function of a continuous random variable like X with parameters μ and σ^2 is:

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < +\infty$$

We can say that variable is normally distributed parameters μ and σ^2 are mean and variance of random variable



Standard Normal Distribution

The distribution of normal random variable with mean 0 and variance 1 is called a standard normal distribution (Z).

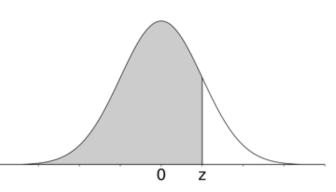
$$Z = \frac{X - \mu}{\sigma} \rightarrow f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} - \infty < Z < +\infty$$

A **standard normal table**, also called the **unit normal table** or **Z table**, is a mathematical table for the values of Φ , which are the values of the cumulative distribution function of the normal distribution. It is used to find the probability that a statistic is observed below, above, or between values on the standard normal distribution, and by extension, any normal distribution. Since probability tables cannot be printed for every normal distribution, as there are an infinite variety of normal distributions, it is common practice to convert a normal to a standard normal and then use the standard normal table to find probabilities.

$$p(Z < z) = \varphi(z)$$

$$p(Z > z) = 1 - p(Z < z)$$

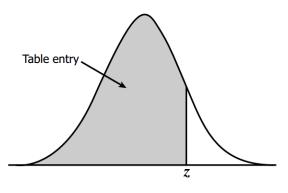
Table of Standard Normal Probabilities for Positive Z-Scores



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
11	8643	8665	8686	8708	8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441

 $p(z < 1.00) = \varphi(1) = 0.8413$

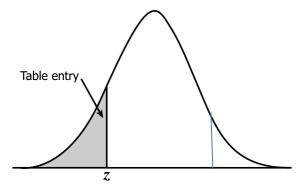
 $p(z > 1.25) = 1 - p(z < 1.25) = 1 - \varphi(1.25) = 1 - 0.8944 = 0.1056$



398 .54 793 .58 179 .62 554 .65 915 .69 257 .72 580 .76 881 .79 159 .81 413 .84 543 .86 349 .88	438 .5 332 .5 217 .6 591 .6 950 .6 951 .7 511 .7 910 .7 186 .8 438 .8	478 . 871 . 255 . 628 . 985 . 324 . 642 . 939 . 212 .	5517 5910 6293 6664 7019 7357 7673 7967	.5557 .5948 .6331 .6700 .7054 .7389 .7704	.5596 .5987 .6368 .6736 .7088 .7422 .7734	.5636 .6026 .6406 .6772 .7123 .7454	.5675 .6064 .6443 .6808 .7157 .7486	.5319 .5714 .6103 .6480 .6844 .7190 .7517 .7823	.5359 .5753 .6141 .6517 .6879 .7224 .7549
793 .58 179 .62 554 .69 555 .69 257 .72 580 .76 381 .79 159 .81 413 .84 543 .86 349 .88	332 .5 217 .6 591 .6 950 .6 951 .7 511 .7 910 .7 186 .8 438 .8	871 . 255 . 628 . 985 . 324 . 642 . 939 . 212 .	5910 6293 6664 7019 7357 7673 7967	.5948 .6331 .6700 .7054 .7389 .7704	.5987 .6368 .6736 .7088 .7422 .7734	.6026 .6406 .6772 .7123 .7454	.6064 .6443 .6808 .7157 .7486	.6103 .6480 .6844 .7190 .7517	.6141 .6517 .6879 .7224 .7549
179 .62 554 .65 915 .69 9257 .72 580 .76 381 .79 159 .81 413 .84 543 .86 349 .88	217 .6 591 .6 950 .6 291 .7 511 .7 910 .7 186 .8 438 .8	255 . 628 . 985 . 324 . 642 . 939 . 212 .	6293 6664 7019 7357 7673 7967	.6331 .6700 .7054 .7389 .7704	.6368 .6736 .7088 .7422 .7734	.6406 .6772 .7123 .7454	.6443 .6808 .7157 .7486	.6480 .6844 .7190 .7517	.6517 .6879 .7224 .7549
554 .65 915 .69 257 .72 580 .76 581 .79 159 .81 413 .84 543 .86 349 .88	591 .6 950 .6 291 .7 511 .7 910 .7 186 .8 138 .8	628 . 985 . 324 . 642 . 939 . 212 .	6664 7019 7357 7673 7967	.6700 .7054 .7389 .7704	.6736 .7088 .7422 .7734	.6772 .7123 .7454	.6808 .7157 .7486	.6844 .7190 .7517	.6879 .7224 .7549
915 .69 257 .72 580 .76 381 .79 159 .81 413 .84 543 .86 349 .88	950 .6 291 .7 511 .7 910 .7 186 .8 438 .8	985 . 324 . 642 . 939 . 212 .	7019 7357 7673 79 <mark>6</mark> 7	.7054 .7389 .7704	.7088 .7422 .7734	.7123 .7454	.7157 .7486	.7190 .7517	.7224 .7549
257 .72 580 .76 581 .79 159 .81 413 .84 543 .86 543 .86	291 .7 511 .7 910 .7 186 .8 438 .8	324 . 642 . 939 . 212 .	7357 7673 79 <mark>67</mark>	.7389 .7704	.7422 .7734	.7454	.7486	.7517	.7549
580 .76 581 .79 159 .81 113 .84 543 .86 543 .86	511 .7 910 .7 186 .8 138 .8	642 . 939 . 212 .	7673 7967	.7704	.7734				
881 .79 159 .81 113 .84 543 .86 849 .88	010 .7 186 .8 138 .8	939 . 212 ·	7967			.7764	.7794	7823	
159 .81 113 .84 543 .86 849 .88	186 .8 138 .8	212 .		.7995	0000				.7852
413 .84 543 .86 349 .88	438 .8		8738		.8023	.8051	.8078	.8106	.8133
413 .84 543 .86 349 .88	438 .8		0200	.8264	.8289	.8315	.8340	.8365	.8389
.88	CE Q	461 .	8485	.8508	.8531	.8554	.8577	.8599	.8621
.88	0. 00	686 .	8708	.8729	.8749	.8770	.8790	.8810	.8830
						.8962	.8980	.8997	.9015
.90							.9147	.9162	.9177
							.9292	.9306	.9319
							.9418	.9429	.9441
							.9525	.9535	.9545
			and the second se				.9616	.9625	.9633
						.9686	.9693	.9699	.9706
									.9767
		05M2022							.9817
		(N. N. S. S.							.9857
									.9890
	and the second se							And the second se	.9916
									.9936
2 (S)									.9952
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10 11 11 11 11 11 11 11 11 11 11 11 11 1			and the second sec	A AND					.9974
									.9981
									.9986
									.99900
									.9993
							and the second second		.9995
200100000 STATE	11 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1								.9997
									.9998
	72 .97 21 .98 61 .98 93 .98 18 .99 38 .99 53 .99 65 .99 74 .99 87 .99 93 .99 93 .99 95 .99	72 .9778 .9 21 .9826 .9 61 .9864 .9 93 .9896 .9 18 .9920 .9 38 .9940 .9 53 .9955 .9 65 .9966 .9 74 .9975 .9 87 .9987 .9 90 .9991 .9 93 .9993 .9 95 .9995 .9	72 .9778 .9783 . 21 .9826 .9830 . 61 .9864 .9868 . 93 .9896 .9898 . 18 .9920 .9922 . 38 .9940 .9941 . 53 .9955 .9956 . 65 .9966 .9967 . 74 .9975 .9976 . 81 .9982 .9987 . 90 .9991 .9991 . 93 .9993 .9994 . 95 .9995 .9995 .	72 .9778 .9783 .9788 21 .9826 .9830 .9834 61 .9864 .9868 .9871 93 .9896 .9898 .9901 18 .9920 .9922 .9925 38 .9940 .9941 .9943 53 .9955 .9956 .9957 65 .9966 .9967 .9968 74 .9975 .9976 .9977 81 .9987 .9987 .9988 90 .9991 .9991 .9991 93 .9993 .9994 .9994 95 .9955 .9955 .9956	72 .9778 .9783 .9788 .9793 21 .9826 .9830 .9834 .9838 61 .9864 .9868 .9871 .9875 93 .9896 .9898 .9901 .9904 18 .9920 .9922 .9925 .9927 38 .9940 .9941 .9943 .9945 53 .9955 .9956 .9957 .9959 65 .9966 .9967 .9968 .9969 74 .9975 .9976 .9977 .9977 81 .9987 .9987 .9988 .9988 90 .9991 .9991 .9992 93 .9993 .9994 .9994 .9994 95 .9995 .9995 .9996 .9996	72.9778.9783.9788.9793.979821.9826.9830.9834.9838.984261.9864.9868.9871.9875.987893.9896.9898.9901.9904.990618.9920.9922.9925.9927.992938.9940.9941.9943.9945.994653.9955.9956.9957.9959.996065.9966.9967.9968.9969.997074.9975.9976.9977.997881.9982.9982.9983.9984.998487.9987.9991.9991.9992.999293.9993.9994.9994.9994.999495.9955.9956.9996.9996.9996	72.9778.9783.9788.9793.9798.980321.9826.9830.9834.9838.9842.984661.9864.9868.9871.9875.9878.988193.9896.9898.9901.9904.9906.990918.9920.9922.9925.9927.9929.993138.9940.9941.9943.9945.9946.994853.9955.9956.9957.9959.9960.996165.9966.9967.9968.9969.9970.997174.9975.9976.9977.9978.997981.9982.9982.9983.9984.998587.9987.9987.9988.9989.998990.9991.9991.9992.9992.999293.9993.9994.9994.9994.999495.9995.9996.9996.9996.9996	72.9778.9783.9788.9793.9798.9803.980821.9826.9830.9834.9838.9842.9846.985061.9864.9868.9871.9875.9878.9881.988493.9896.9898.9901.9904.9906.9909.991118.9920.9922.9925.9927.9929.9931.993238.9940.9941.9943.9945.9946.9948.994953.9955.9956.9957.9959.9960.9961.996265.9966.9967.9968.9969.9971.997274.9975.9976.9977.9978.9979.997981.9982.9982.9983.9984.9984.9985.998587.9987.9987.9988.9988.9989.9989.998990.9991.9991.9994.9994.9994.9994.999595.9995.9996.9996.9996.9996.9996.9996	72.9778.9783.9788.9793.9798.9803.9808.981221.9826.9830.9834.9838.9842.9846.9850.985461.9864.9868.9871.9875.9878.9881.9884.988793.9896.9898.9901.9904.9906.9909.9911.991318.9920.9922.9925.9927.9929.9931.9932.993438.9940.9941.9943.9945.9946.9948.9949.995153.9955.9956.9957.9959.9960.9961.9962.996365.9966.9967.9968.9970.9971.9972.997374.9975.9976.9977.9978.9979.998081.9982.9982.9983.9984.9985.9985.998687.9987.9987.9988.9989.9989.999090.9991.9991.9991.9992.9992.9992.999393.9993.9994.9994.9994.9994.9995.999595.9995.9996.9996.9996.9996.9996.9996.9996

 $p(z \leq 2.20) = \varphi(2.2) = 0.9861$

 $p(z > 0.25) = 1 - p(z < 0.25) = 1 - \varphi(0.25) = 1 - 0.5987 = 0.4013$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

 $p(z \le -0.64) = \varphi(-0.64) = 0.2611$

$$p(z > -1.95) = 1 - p(z < -1.95) = 1 - \varphi(-1.95) = 1 - 0.0256 = 0.9744$$

$$p(z > -3.55) = 1 - p(z < -3.55) = 1 - 0.0000 \approx 1$$

Example: a certain type of storage battery lasts, on average, 3 years with standard deviation of 0.5 year. Assume that the battery lives are normally distributed. Find

- Probability that a given battery will last less than 2.3 years
- Probability that a given battery will last more than or equal 3.3 years

$$p(x < 2.3) = p\left(\frac{x-\mu}{\sigma} < \frac{2.3-3}{0.5}\right) = p(z < -1.4) = \varphi(-1.4) = 0.0808$$
$$p(x > 3.3) = 1 - p(x < 3.3) = 1 - p\left(\frac{x-\mu}{\sigma} < \frac{3.3-3}{0.5}\right) = 1 - p(z < 0.6)$$
$$= 1 - 0.7257 = 0.2743$$

Example: an electrical firm produces light bulbs that their lives before burn-out, are normally distributed with mean $\mu = 800 \ hrs$ and variance $\sigma^2 = 1600 \ hrs$. Find

- Probability that a bulb burn between 778 and 834 hours

$$p(778 < x < 834) = p\left(\frac{778 - 800}{40} < \frac{x - \mu}{\sigma} < \frac{834 - 800}{40}\right)$$
$$= p(-0.55 < z < 0.85) = p(z < 0.85) - p(z < -0.55)$$
$$= 0.8023 - 0.2912 = 0.5114$$

Example: an electrical firm produces two type of light bulbs that their lives before burn-out, are normally distributed with mean $\mu = 800 \& 700 hrs$ and variance $\sigma^2 = 1600 \& 1400 hrs$. Find

- Probability that total life of two bulbs will exceed 1700 hours
- Probability that difference life of two bulbs will not exceed 50 hours

$$p(x+y>1700) = \left(\frac{(x+y) - \mu_{x+y}}{\sigma_{x+y}} > \frac{1700 - 1500}{\sqrt{3000}}\right) = p\left(z > \frac{200}{54.77}\right)$$
$$= p(z>3.65) = 1 - p(z<3.65) = 1 - 1 = 0$$

$$\mu_{x+y} = \mu_x + \mu_y = 800 + 700 = 1500$$

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 = 1600 + 1400 = 3000$$

$$p(x - y < 50) = \left(\frac{(x - y) - \mu_{x - y}}{\sigma_{x - y}} < \frac{50 - 100}{\sqrt{3000}}\right) = p\left(z < \frac{050}{54.77}\right)$$
$$= p(z < -0.91) = 0.1814$$

$$\mu_{x-y} = \mu_x - \mu_y = 800 - 700 = 100$$

$$\sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 = 1600 + 1400 = 3000$$

Central Limit Theorem:

p

The sum of large number of independent random variables is approximately normally distributed.

Let $x_1, x_2, ..., x_n$ be a sample from a population having $mean = \mu$ and $sd = \sigma$.

For *n* large, the sum of $x_1 + x_2 + \cdots + x_n$ will approximately have a normal distribution with mean $n\mu$ and $sd = \sigma\sqrt{n}$.

Example: An assurance company has 10,000 automobile policy holders, if the expected yearly claim per policy holder is 260\$ with standard deviation of 800\$, approximate the probability that total yearly claim exceeds 2.8\$ million.

$$y = \sum_{i=1}^{10000} x_i \sim N(n\mu, \sigma\sqrt{n}) \sim N(260 * 10^4, 8 \times 10^4)$$
$$(y > 2.8 \times 10^6) = p\left(\frac{y - \bar{y}}{\sigma_y} > \frac{2.8 \times 10^6 - 2.6 \times 10^6}{8 \times 10^4}\right) = p(z > 2.5) = 1 - p(z < 2.5)$$
$$= 1 - 0.9938 = 0.0062$$

When n is large (n > 30) then $\overline{x} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$

Example: The blood cholesterol level of a population of workers has mean equal 202 and standard deviation equal 14. If a sample of 36 workers is selected, approximate the probability that sample mean of their blood cholesterol level will be between 198 and 206.

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \sim N(202, \frac{14}{\sqrt{36}})$$

$$p(198 < \bar{x} < 206) = p\left(\frac{198 - 202}{14/\sqrt{36}} < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{206 - 202}{14/\sqrt{36}}\right) = p(-1.71 < z < 1.71)$$

$$= p(z < 1.71) - p(z < -1.71) = 0.9564 - 0.0436 = 0.9128$$

Note 1: for central limit theorem $n \ge 30$ is enough

Note 2 : if the underlying population distribution is normal, then the sample mean \bar{x} will also be normal. No matter what the sample size is.