E.M.U. - FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS

MATH 106 -- Linear Algebra -- Final Examination
28th of May 2011

Duration: 2 hours.

Name-Surname		Student no.		Signature		Group
Q1	Q2	Q3	Q4	Q5	Q6	Total;

Remark: Questions 4 and 6 are also Quiz 4 questions.
Each question is worth 20 points.

Q1) Use Gaussian elimination to determine the solution set to the given linear system

$$\begin{aligned} x + 2y + z &= 1 \\ 3x + 5y + z &= 3 \\ 2x + 6y + 7z &= 1 \end{aligned}$$

Augmented matrix is $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right) \xrightarrow[R_3 \to -2R_1+R_3]{R_2 \to -3R_1+R_2}$

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right) (2)R_2 + R_3 \to R_3 \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$

$\xrightarrow{(-1)R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$ System has unique solutions

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using backward substitution

$z = -1$

$y + 2z = 0 \Rightarrow y + 2(-1) = 0 \Rightarrow y = 2$

$x + 2y + z = 1 \Rightarrow x = 1 - 2y - z$

$x = 1 - 2(2) - (-1)$

$x = 1 - 4 + 1$ 10

$x = -2$

Q2) Evaluate the following **determinant;**

$$\begin{vmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_4} = (-1) \begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 3 & 5 & -1 & 2 \end{vmatrix}$$

$R_2 \to -2R_1 + R_2$
$R_3 \to -3R_1 + R_3$
$R_4 \to -3R_1 + R_4$

no change in determinant

$$= (-1) \begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & -1 & 4 \\ 0 & 8 & -7 & -1 \end{vmatrix} = (-1)(1) \begin{vmatrix} 3 & 1 & 0 \\ 5 & -1 & 4 \\ 8 & -7 & -1 \end{vmatrix}$$

$$= -1 \left[+3 \begin{vmatrix} -1 & 4 \\ -7 & -1 \end{vmatrix} - 1 \begin{vmatrix} 5 & 4 \\ 8 & -1 \end{vmatrix} + 0 \right]$$

$$= -1 \left[3(1 + 28) - 1(-5 - 32) \right]$$

$$= -[87 + 37]$$

$$= -124$$

Q3) a) Find the standard matrix for the linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$ that first rotates a vector anticlockwise about the x-axis through an angle of 60°, and then reflects the resulting vector about the xz-plane.

b) Find the image of the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ under this transformation.

a) $$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60 \\ 0 & \sin 60 & \cos 60 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad S$$

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S$$

$$[T_2 \circ T_1] = [T_2] \cdot [T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad S$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1+\frac{3\sqrt{3}}{2} \\ \sqrt{3}+\frac{3}{2} \end{bmatrix} \quad S$$

Q4) Show that the vectors $\mathbf{v}_1 = (-1,3,2)$, $\mathbf{v}_2 = (1,-2,1)$, $\mathbf{v}_3 = (2,1,1)$ **span** $\mathbb{R}^3$, and express the vector $(3,2,7)$ as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$(b_1,b_2,b_3) = c_1(-1,3,2) + c_2(1,-2,1) + c_3(2,1,1)$.

$$-c_1 + c_2 + 2c_3 = b_1$$
$$3c_1 - 2c_2 + c_3 = b_2$$
$$2c_1 + c_2 + c_3 = b_3$$

$$\begin{pmatrix} -1 & 1 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad A\underline{c} = \underline{b}$$

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$A\underline{c} = \underline{b}$ is consistent for all $\underline{b}$ iff A is invertible.

$\Leftrightarrow$ iff $\det A \neq 0$.

$$|A| = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -1(-3) - 1(3-2) + 2(3+4) = 3 - 1 + 14 = 16 \neq 0.$$

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So, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ span $\mathbb{R}^3$.

$k_1(-1,3,2) + k_2(1,-2,1) + k_3(2,1,1) = (3,2,7)$

$$-k_1 + k_2 + 2k_3 = 3.$$
$$3k_1 - 2k_2 + k_3 = 2$$
$$2k_1 + k_2 + k_3 = 7.$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 3 \\ 3 & -2 & 1 & 2 \\ 2 & 1 & 1 & 7 \end{array}\right) \underset{\sim}{\overset{1}{R_1 \to -R_1}} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 3 & -2 & 1 & 2 \\ 2 & 1 & 1 & 7 \end{array}\right) \begin{array}{c} \overset{2}{R_2 \to -3R_1 + R_2} \\ \sim \\ \underset{2}{R_3 \to -2R_1 + R_3} \end{array} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 7 & 11 \\ 0 & 3 & 5 & 13 \end{array}\right)$$

$$\underset{\sim}{\overset{2}{R_3 \to -3R_2 + R_3}} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -16 & -20 \end{array}\right)$$

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$k_3 = \frac{20}{16} = \frac{5}{4}$ 1

$k_2 + 7 \cdot \frac{5}{4} = 11$ 1

$k_2 = \frac{44 - 35}{4} = \frac{9}{4}$

$k_1 - 9/4 - \frac{10}{4} = -3$

$k_1 = -12 + 19/4 = 7/4$ 1

Q5) Use the **Wronskian method** to show that $f_1 = 1,\ f_2 = x,\ f_3 = x^2$ form a linearly independent set of vectors in $C^2(-\infty,\infty)$.

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2. \end{vmatrix} \underset{\text{along 1st col.}}{\overset{\text{Cof. exp.}}{=}} 2.$$

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not zero

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So these functions are L.I.

4.

Q6) a) Determine whether the set $\left\{\begin{pmatrix}3 & 6\\ 3 & -6\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -8\\ -12 & -4\end{pmatrix}, \begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}\right\}$ is **linearly independent** in M_{22}.

b) Determine whether this set forms a **basis** for M_{22}.

a)

$$c_1\begin{pmatrix}3 & 6\\ 3 & -6\end{pmatrix} + c_2\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} + c_3\begin{pmatrix}0 & -8\\ -12 & -4\end{pmatrix} + c_4\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\left.\begin{aligned} 3c_1 + c_4 &= 0\\ 6c_1 - c_2 - 8c_3 &= 0\\ 3c_1 - c_2 - 12c_3 - c_4 &= 0\\ -6c_1 - 4c_3 + 2c_4 &= 0\end{aligned}\right\} \quad \begin{array}{c} 3 \quad 6 \quad 3 \quad -6 \\ \begin{pmatrix}3 & 0 & 0 & 1\\ 6 & -1 & -8 & 0\\ 3 & -1 & -12 & -1\\ -6 & 0 & -4 & 2\end{pmatrix}\end{array}\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

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$$A\underline{c} = \underline{0}$$

Check det. of A; if nonzero, then A^{-1} exists and $\underline{c} = \underline{0}$ is the only sol^n.

$$\begin{vmatrix}3 & 0 & 0 & 1\\ 6 & -1 & -8 & 0\\ 3 & -1 & -12 & -1\\ -6 & 0 & -4 & 2\end{vmatrix} = 3\begin{vmatrix}-1 & -8 & 0\\ -1 & -12 & -1\\ 0 & -4 & 2\end{vmatrix} - 1\begin{vmatrix}6 & -1 & -8\\ 3 & -1 & -12\\ -6 & 0 & -4\end{vmatrix}$$

$$= 3\{-1(-28) + 8(-2)\} - 1\{1(-84) - 1(-72)\}$$

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$$= 3\{\cancel{28-16}\ 12\} - 1\{-12\} = 36 + 12 = 48 \neq 0.$$

So $\underline{c} = \underline{0}$ is the only sol^n. Hence this set is L. I.

b) $\dim(M_{22}) = 4$ and the set in part (a) contains 4 L. I. matrices. So this set is a basis for M_{22}.

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