**Graphs**

**What is a Graph?**

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Informal definition:

* A *graph* is a mathematical abstraction used to represent "connectivity information".
* A graph consists of *vertices* and *edges* that connect them, e.g.,

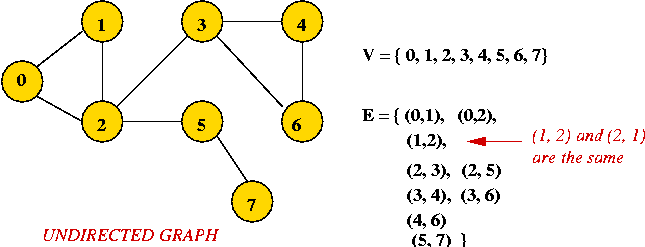


* It shouldn't be confused with the "bar-chart" or "curve" type of graph.

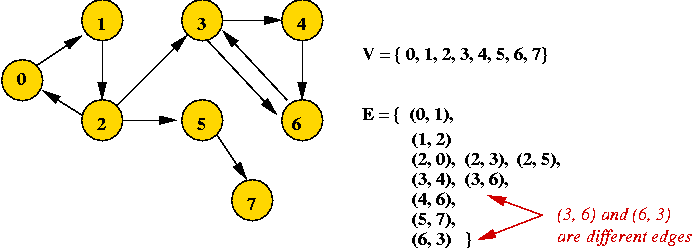
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Formally:

* A graph *G = (V, E)* is:
  + a set of vertices *V*
  + and a set of edges *E = { (u, v): u* and *v* are vertices }.
* Two types of graphs:
  + **Undirected graphs**: the edges have no direction.
  + **Directed graphs**: the edges have direction.
* Example: undirected graph



* + Edges have no direction.
  + If an edge connects vertices *1* and *2*, either convention can be used:
    - No duplication: only one of *(1, 2)* or *(2, 1)* is allowed in *E*.
    - Full duplication: both *(1, 2)* and *(2, 1)* should be in *E*.
* Example: directed graph

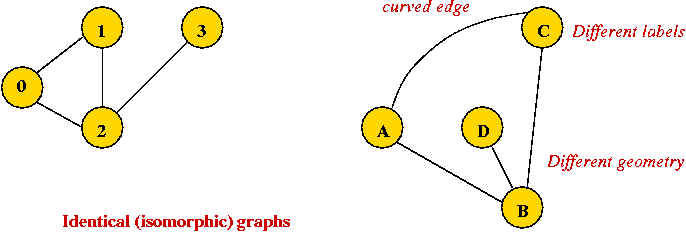


* + Edges have direction (shown by arrows).
  + The edge *(3, 6)* is not the same as the edge *(6, 3)* (both exist above).

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Depicting a graph:

* The picture with circles (vertices) and lines (edges) is *only a depiction*   
  => a graph is purely a mathematical abstraction.
* Vertex labels:
  + Can use letters, numbers or anything else.
  + Convention: use integers starting from 0.   
    => useful in programming, e.g. degree[i] = degree of vertex i.
* Edges can be drawn "straight" or "curved".
* The geometry of drawing has no particular meaning:

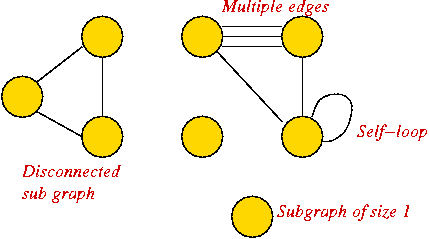


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Graph conventions:

* What's allowed (but unusual) in graphs:
  + Self-loops (occasionally used).
  + Multiple edges between a pair of vertices (rare).
  + Disconnected pieces (frequent in some applications).

Example:

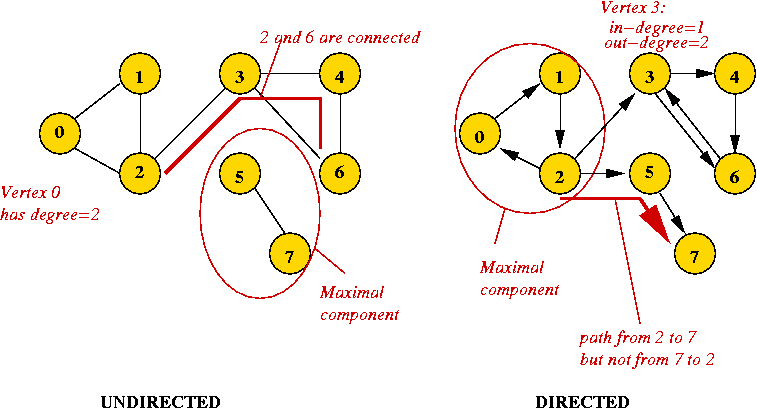


* What's not (conventionally) allowed:
  + Mixing undirected and directed edges.
  + Re-using labels in vertices.
  + Bidirectional arrows.
* Most common:
  + No multiple edges.
  + No self-loops.
* Other terms used:
  + Vertices: nodes, terminals, endpoints.
  + Edges: links, arcs.

Definitions:

* **Degrees**:
  + Undirected graph: the **degree** of a vertex is the number of edges incident to it.
  + Directed graph: the **out-degree** is the number of (directed) edges leading out, and the **in-degree** is the number of (directed) edges terminating at the vertex.
* **Neighbors**:
  + Two vertices are **neighbors** (or are **adjacent**) if there's an edge between them.
  + Two edges are **neighbors** (or are **adjacent**) if they share a vertex as an endpoint.
* **Paths**:
  + Undirected: a sequence of vertices in which successive vertices are adjacent.
  + Directed: a sequence of vertices in which every pair of successive vertices has this property: there's a directed edge from the first to the second.
  + A **simple path** does not repeat any vertices (and therefore edges) in the sequence.
  + A **cycle** is a simple path with the same vertex as the first and last vertex in the sequence.
* **Connectivity:**
  + Undirected: Two vertices are **connected** if there is a path that includes them.
  + Directed: Two vertices are **strongly-connected** if there is a (directed) path from one to the other.
* **Components:**
  + A **subgraph** is a subset of vertices together with the edges from the original graph that connects vertices in the subset.
  + Undirected: A **connected component** is a subgraph in which every pair of vertices is connected.
  + Directed: A **strongly-connected component** is a subgraph in which every pair of vertices is strongly-connected.
  + A **maximal component** is a connected component that is not a proper subset of another connected component.
* **Digraph**: another name for a *directed graph*.

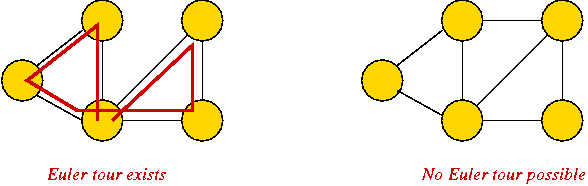
Example:



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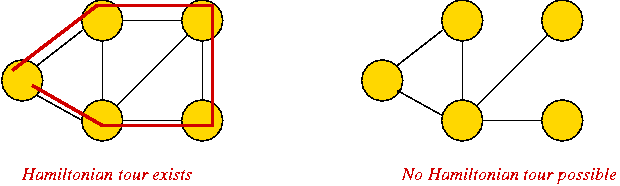
More definitions:

* **Euler tour**: A cycle that traverses all edges exactly once (but may repeat vertices).



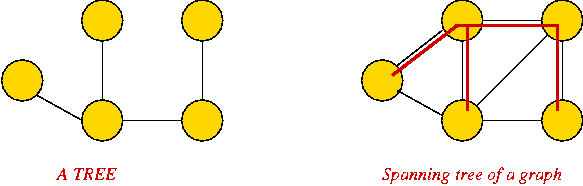
Known result: Euler tour exists if and only if all vertices have even degree.

* **Hamiltonian tour**: A cycle that traverses all vertices exactly once.

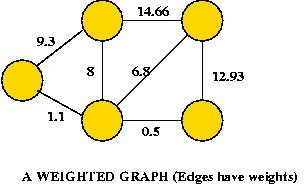


Known result: testing existence of a Hamiltonian tour is (very) difficult.

* **Euler path**: A path that traverses all edges exactly once.
* **Hamiltonian path**: A path that traverses all vertices exactly once.
* **Trees**:
  + A **tree** is a connected graph with no cycles.
  + A **spanning tree** of a graph is a connected subgraph that is a tree.



* **Weighted graphs**:
  + Sometimes, we include a "weight" (number) with each edge.
  + Weight can signify length (for a geometric application) or "importance".
  + Example:



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**Graph Data Structures**

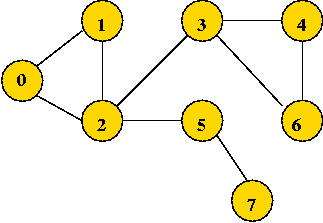
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First, an idea that doesn't work:

* We have already represented trees (like binary trees) with node instances and pointers between instances.
* Idea: use a node instance for each vertex, and a pointer from one vertex to another if an edge exists between them.

The two fundamental data structures:

* Adjacency matrix.
  + Key idea: use a 2D matrix.
  + Row *i* has "neighbor" information about vertex *i*.
  + Undirected: adjMatrix[i][j] = 1 if and only if there's an edge between vertices *i* and *j*.   
    adjMatrix[i][j] = 0 otherwise.
  + Directed: adjMatrix[i][j] = 1 if and only if there's an edge from *i* to *j*.   
    adjMatrix[i][j] = 0 otherwise.
  + Example: undirected



0 **1** **1** 0 0 0 0 0

**1** 0 **1** 0 0 0 0 0

**1** **1** 0 **1** 0 **1** 0 0

0 0 **1** 0 **1** 0 **1** 0

0 0 0 **1** 0 0 **1** 0

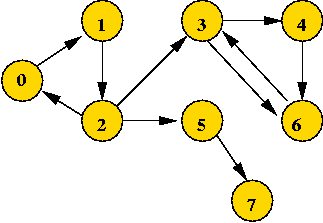
0 0 **1** 0 0 0 **1** **1**

0 0 0 **1** **1** 0 0 0

0 0 0 0 0 **1** 0 0

Note: adjMatrix[i][j] == adjMatrix[j][i] (convention for undirected graphs).

* + Example: directed



0 **1** 0 0 0 0 0 0

0 0 **1** 0 0 0 0 0

**1** 0 0 **1** 0 **1** 0 0

0 0 0 0 **1** 0 **1** 0

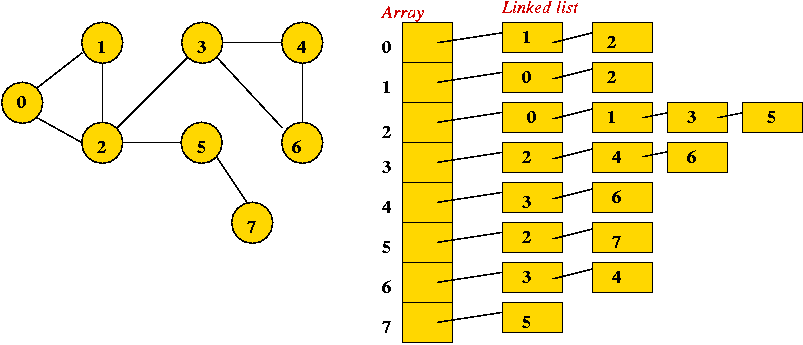
0 0 0 0 0 0 **1** 0

0 0 **1** 0 0 0 0 **1**

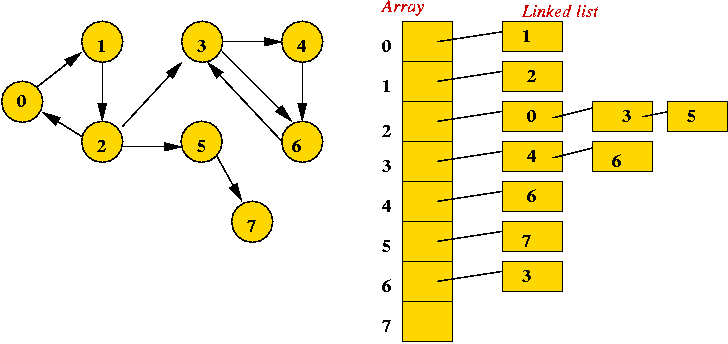
0 0 0 **1** 0 0 0 0

0 0 0 0 0 0 0 0

* Adjacency list.
  + Key idea: use an array of vertex-lists.
  + Each vertex list is a list of neighbors.
  + Example: undirected



* + Example: directed



* + Convention: in each list, keep vertices in order of insertion   
    => add to rear of list
* Both representations allow complete construction of the graph.
* Advantages of matrix:
  + Simple to program.
  + Some matrix operations (multiplication) are useful in some applications (connectivity).
  + Efficient for dense (lots of edges) graphs.
* Advantages of adjacency list:
  + Less storage for sparse (few edges) graphs.
  + Easy to store additional information in the data structure.   
    (e.g., vertex degree, edge weight)

**Breadth-First Search**

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About graph search:

* "Searching" here means "exploring" a particular graph.
* Searching will help reveal properties of the graph   
  e.g., is the graph connected?
* Usually, the input is: vertex set and edges (in no particular order).

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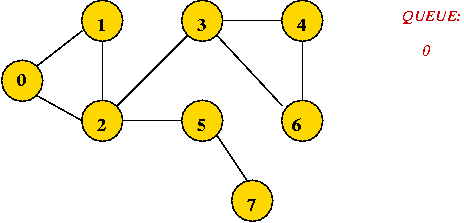
Key ideas in breadth-first search: (undirected)

* Mark all vertices as "unvisited".
* Initialize a queue (to empty).
* Find an unvisited vertex and apply breadth-first search to it.
* In breadth-first search, add the vertex's neighbors to the queue.
* Repeat: extract a vertex from the queue, and add its "unvisited" neighbors to the queue.

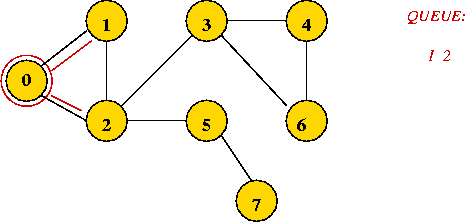
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Example:

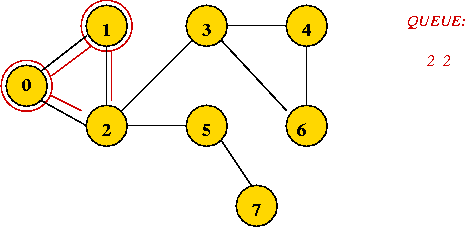
* Initially, place vertex 0 in the queue.



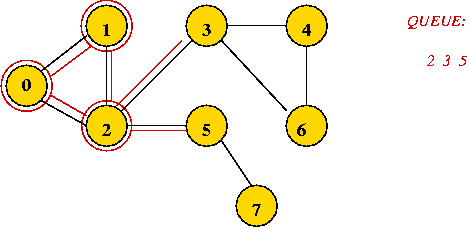
* Dequeue 0   
  => mark it as visited, and add its unvisited neighbors to queue:



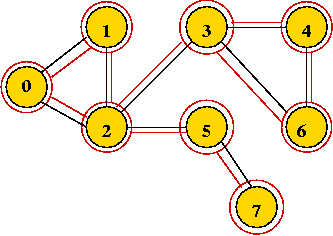
* Dequeue 1   
  => mark it as visited, and add its unvisited neighbors to queue:



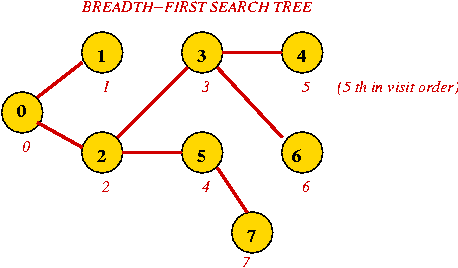
* Dequeue 2   
  => mark it as visited, and add its unvisited neighbors to queue:



* Dequeue 2   
  => it's already visited, so ignore.
* Continuing ...



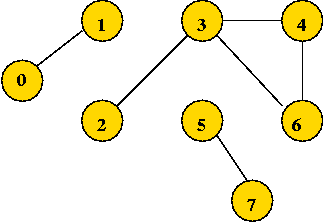
* Breadth-first search tree, and visit order:



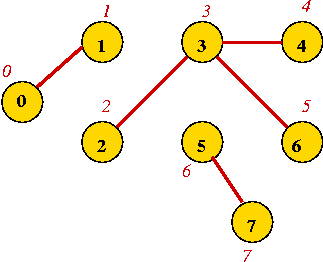
* + Exploring an edge: examining an unvisited neighbor.
  + If an unvisited neighbor gets on the queue for the first time, the edge is called a "tree edge".
  + Putting the tree edges and all vertices together results in: the *breadth-first search tree*.
  + For a particular graph and its implementation, the tree produced is unique.
  + However, starting from another vertex will result in another tree, that may be just as useful.

Searching an unconnected graph:

* The connected components are explored in order:
* Example:



The tree, and visit order:



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Applications:

* Connectivity:
  + Breadth-first search identifies connected components.
  + However, depth-first search is preferred (required for directed graphs).
* Shortest paths:
  + A path between two vertices in the tree is the shortest path in the graph.
* Optimization algorithms:
  + Various problems result in "graph search space".
  + BFS together with "exploration rules" is often used to search for solutions (e.g., branch-and-bound exploration).

Note: BFS works on a weighted graph by ignoring the weights and only using connectivity information (i.e., is there an edge or not?).

**Depth-First Search on Undirected Graphs**

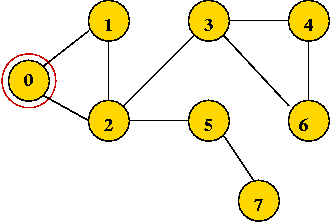
Key ideas:

* Mark all vertices as "unvisited".
* Visit first vertex.
* Recursively visit its "unvisited" neighbors.

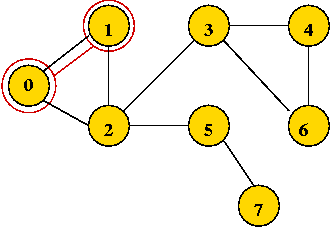
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Example:

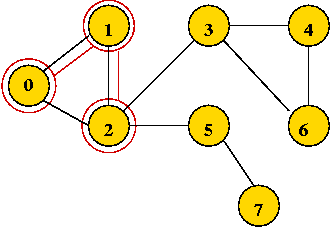
* Start with vertex 0 and mark it visited.



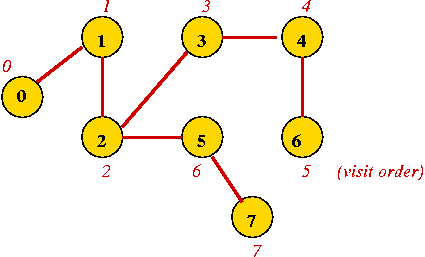
* Visit the first neighbor 1, mark it visited.



* Explore 1's first neighbor, 2.



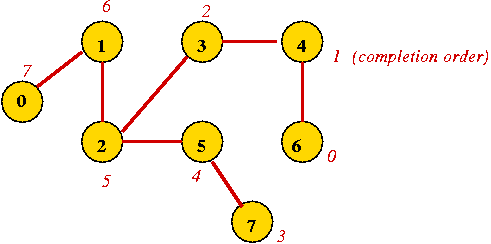
* Continuing until all vertices are visited ...



* + Vertices are marked in order of visit.
  + An edge to an unvisited neighbor that gets visited next is in the depth-first search tree.

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* Completion order:
  + In breadth-first search, once a vertex is processed, it is never processed again.
  + In depth-first, we also encounter a vertex after returning from the recursive call.   
    => we can record a *completion order*.



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**Depth-First Search in Directed Graphs**

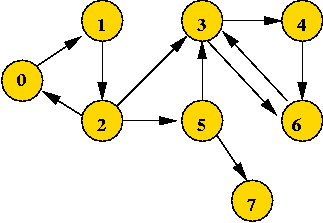
Key ideas:

* A straightforward depth-first search is similar to the undirected version   
  => only explore edges going outward from a vertex in a directed graph.
* In addition to "back" and "down" edges, it is useful to identify "cross" edges.

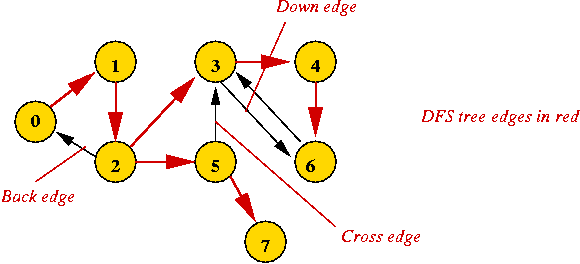
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Example:

* Consider: (slightly different from previous example)



* Applying DFS gives:



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